Model Fitting and Parameter Estimation of Neural Mass Models

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# Contents

1 An Introduction to Neural Mass Models ................................................. 3  
1.1 Introduction ......................................................................................... 3  
1.2 Building Blocks of Neural Mass Models ............................................ 5  
  1.2.1 The Mean Voltage and Activation Function .................................. 5  
  1.2.2 Input to a Neural Mass from Outside Sources .............................. 6  
  1.2.3 The Synaptic Response Function .................................................. 8  
  1.2.4 The Standard Parameters .............................................................. 9  
1.3 A Simple Neural Mass Model ............................................................. 10  
1.4 A Model With Feedback ..................................................................... 11  
  1.4.1 Bifurcations ....................................................................................... 13  
  1.4.2 Bi-stability ...................................................................................... 15  
1.5 Coupled Neural Masses ..................................................................... 18  

2 A Model for Pan-Cortical Activity ......................................................... 23  
2.1 Introduction ......................................................................................... 23  
2.2 A Neural Mass Model ......................................................................... 23  
2.3 Manipulating the Model ..................................................................... 25  
2.4 Manipulating the Data ....................................................................... 30  
2.5 Fitting the Model ............................................................................... 33  
2.6 Estimating Sigma ............................................................................... 40  

Appendices ....................................................................................... 44  

A Linear Time–Invariant Systems .......................................................... 44  

B Supercritical Hopf Bifurcations and Time Change for Local Martingales 46  

C Programming, Plotting Figures and Model Simulations ......................... 48  

D Power Spectral Density ........................................................................ 51  

E Nyquist–Shannon Sampling Theorem and Filters .................................. 55  

Bibliography ...................................................................................... 59
1 An Introduction to Neural Mass Models

1.1 Introduction

It is hard to overemphasise how important brains are for humans - as an organ it is central to our lives and experiences. Attempting to understand the patterns and relationships concerning the voltage fluctuations between different neurons in the brain has been a long-term aim for neuroscientists. This undertaking took a big leap forward in the 20th century, when non-intrusive methods were invented to measure voltage potential activity in the outer regions of the brain.

Two such techniques include Electroencephalography (see e.g. [31]) and Magnetoencephalography (see e.g. [11]), respectively known as EEG and MEG. EEG attempts to measure neuronal activity in the cortex by placing multiple electrodes on the scalp of a subject, whilst MEG uses magnetometers to measure the magnetic fields produced by the cortical region outside of the subject’s head. Readings gathered from both techniques can then be used to infer how underlying voltage potentials in the outer brain changes with respect to time. Hopefully these readings over a period of time might enlighten us on how different regions of the cortex communicate with each other, therefore helping us understand how underlying cognitive processes are implemented dynamically. Whilst recorded voltage fluctuations are typically oscillatory (called brain rhythms), unusual voltage fluctuations can indicate conditions such as schizophrenia, Alzheimer’s disease, epileptic seizures or lack of oxygen in the brain. In this thesis we will often encounter repetitive periodic fluctuations, which signal very unhealthy brain states than can occur in some individuals. We present an example:

![Figure 1: Two typical examples of rhythms:](image1)

(a) An MEG reading where the subject is in a resting state (from [14]) - perfectly healthy subject
(b) An EEG reading where the subject is in a postanoxic coma (from [4]) - a pathological brain state

Whilst much data on these rhythms have been gathered from both humans and other primates and analysed, it is very useful to create and examine models that can simulate cortical voltages. Linking biological parameters and phenomena with aspects of our model, we hope that examining how different areas of our model contribute to varying model-derived or simulated results, will imply that there is a certain type of interplay between the biological components. Mathematical descriptions of the model
behaviour can then help us think about the brain in a more theoretical manner and allow us to classify different types of voltage dynamics via mathematical concepts. This may then allow us to discuss macroscopic and time-dependant brain activity in more detail.

Therefore for our model to be useful, it need not be a perfect model of cortical dynamics nor generate data with an (almost) perfect statistical structure as reality. The model just needs to produce stylistically similar global dynamics to a real-life cortex and capture the right physiological mechanisms. Since its parameters should be related to real-life biological variables, tweaking them and observing this effect on results would then suggest underlying biological causes of brain phenomena.

Whilst the popular saying that “one has more connections in a cubic millimetre of one’s brain than stars in the Milky Way” is false [1], there are still millions of neurons in the cortex or outer brain. As each neuron would be described in a model as an interaction of multiple parameters, trying to incorporate a perfect or highly accurate description of neuronal dynamics into our model produces two problems: 1) It would be very difficult to conduct simulations or calculate theoretical properties of the model due to the necessity of large computational resources 2) Such a model would be so complicated due to numerous parameters, that it may not provide us with any biological insight. We tackle these problems by noting the following fact: EEG/MEG measures the voltages of regions of neurons - called neural masses, permitting us to see how different regions of the brain communicate with each other. Therefore we can greatly reduce the dimensionality of our model by using what we call a neural mass model. This involves modelling clumps of neurons with their average voltage, along with the interactions between these different masses. As in reality the local voltage within cortical tissue is proportional to the measured scalp potential [31], we can model the hypothetical EEG/MEG voltage, observed to be proportional to this average neural mass voltage produced by our model. The general approach to constructing a model is to define a biophysical model of local cortical dynamics. Multiple local masses will then be coupled to each other in some manner to simulate pan-cortical brain activity.

As previously stated, an advantage of using mathematical models to mimic brain activity, is that it is easy to link model behaviour with mathematical results. This is especially useful in our case, because neurons are dynamical and respond to stimuli differently depending on their past activity. Therefore we need to consider neuronal voltages as being part of a macroscopic dynamical system. This signifies the “state” of a neural mass will not only be its present voltage, or an input-output response, but a macroscopic picture of how all neuronal dynamics change over a short period of time. Elements of the theory of dynamical systems such as bifurcation theory, can suggest how global neuronal dynamics might change. We have already presented an example of this in figure-1b, which would be described as a limit cycle. An excellent introduction to the relationship between neuroscience and dynamical systems is [18].

Another advantage of using a model, is that we can try and incorporate and analyse the effect of random inputs into the cortex, by using the mathematical field of stochastic analysis. These inputs can either come from sub-cortical structures (being so complicated we treat them as being random), or simply from the inherent randomness of brain dynamics. We model these random inputs as “noise”, a process that is done throughout the scientific world [15]. As these random input models are somehow
related in a way to concepts studied in stochastic analysis, we can use some tools developed in that field to understand how this might affect real life results. This addition of probabilistic processes in the system, adds an extra richness to the deterministic results obtained by standard non-random dynamical models and can sometimes influence global transitional properties of voltage fluctuations. In fact, noise and randomness in the brain are a crucial component of how it functions [36].

In this section, we introduce the components that make up these neural mass models, which simulate voltage potential of regions of the cortex. We start by examining how we may model the voltage and firing rate of each individual neural mass. We then discuss how we might model the interactions between these masses. Then we consider three simple neural mass models and start to observe interesting properties of global dynamics that they fulfil.

In section-2, we will extend these concepts to discuss a neural mass model for the entire cortex, that is used in current research.

1.2 Building Blocks of Neural Mass Models

We describe some concepts, models and examples introduced in [13]. If the reader is searching for a biological introduction to concepts in neuroscience to explain the following processes, one could start with the texts [5],[18], or consider the more advanced text [25]. We immediately confess to the reader that these are gross simplifications of very complicated biological processes, however even simplistic definitions can produce a useful model.

1.2.1 The Mean Voltage and Activation Function

For a neural mass denoted $x$, it seems intuitive to define the average voltage of all the neurons in the mass at time $t$ as

$$V_x(t) \quad (1)$$

This voltage potential is created by multiple neurons having different concentrations of charged particles on either side of membranes.

Neurons and neural masses also send signals to each other via voltage spikes. These voltage spikes are themselves functions of the current (mean) mass voltages and not at all dependent on previous voltages, strictly before the present time. Therefore it seems plausible to define a function $S$, that takes as an input the mean current voltage of a neural mass and outputs an average firing rate (in possibly multiple directions to multiple neural masses). We denote this as:

$$S(V_x(t)) \quad (2)$$

As a very small change in mass voltage, will result in a very small change of the output firing rate, we find it reasonable to let $S(\cdot) \in C^1$. Additionally, a neural mass having a higher voltage would also have a higher firing rate, so we let

$$S'(\cdot) > 0 \quad (3)$$

Due to the fact that neurons have refractory periods; an incapability to fire continuously during a period of time, the firing rate must be bounded

$$\lim_{V \rightarrow -\infty} S(V) =: Q_{\text{max}} < \infty, \quad 0 = \lim_{V \rightarrow -\infty} S(V) \quad (4)$$
here $Q_{\text{max}}$ corresponds to the maximal possible firing rate of a neural mass. A commonly used function for $S$ satisfying equations 1, 2, 3, 4, is called the Sigmoid function:

$$S(V) := \frac{Q_{\text{max}}}{1 + e^{-\frac{V-\theta}{\sigma}}}; \quad \sigma, \theta > 0$$

(5)

Slight variations of this function are ubiquitous in computational neuroscience. Also note $S(\theta) = \frac{Q_{\text{max}}}{2}$ is half of the maximal firing rate and $S'(\cdot)$ is symmetric about $\theta$. $\sigma$ is related to the dispersion of membrane potentials over the neuronal population in a mass. We draw an example of this function:

![Figure 2: A drawing of the sigmoid function as in equation-5, where we take $\theta = 0.015$, $\sigma = 0.003$, $Q_{\text{max}} = 250.0$ (the values we will use for this section)](image)

Throughout this thesis we use the sigmoid function presented in equation-5, with the parameters presented in figure-2.

### 1.2.2 Input to a Neural Mass from Outside Sources

Within a neural mass model, we will also have inputs from sources that are not within the collection of neural masses that we consider - called external inputs. We broadly divide these into two types, fixed and probabilistic. The fact that neurons and neural masses have random inputs is fundamental to how we analyse signals and essential to the human experience [36]. A general way of describing this input for neural mass $x$ is:

$$Q_{\text{in}}^x(t) = q^x + \delta^x \xi(t) \quad q, \delta \geq 0$$

(6)

where the term $q$ describes a fixed constant input into the neural mass and the term $\delta \xi(t)$ describes the probabilistic input into the neural mass. $q$ is just a positive real number, whilst the much more complicated probabilistic part consists of standard white noise $\xi(t)$ with “variance” $\delta$. Since a neural mass receives so many (partially) independent inputs, it seems reasonable to model $Q_{\text{in}}$ as some sort of Gaussian (via the central limit theorem). Modelling random inputs as white noise is ubiquitous in engineering and physics (see e.g [15]), but mathematically it is a very complicated object. Here we want to provide some sort of explanation of how we are going to perform calculations with this white noise. As astutely observed in [26], white noise can have different meanings depending on who is the interpreter:
1) **Musician** A sound with equal intensity, for any frequency within a very broad band.

2) **Engineer** A noise that disturbs an input, so that the output of a system is a function of both the original input and the noise. This is a random process $\xi(t)$ depending on time, with mean 0 and infinite variance, where $t_1 \neq t_2$ implies that $\xi(t_1) \perp \xi(t_2)$. Additionally $\mathbb{E}[(\int_0^t f(s)\xi(s)\, ds)^2] = \int_0^t f(s)^2\, ds$.

3) **Mathematician** The derivative of Brownian motion with respect to time.

Whichever definition you take, it is clear that white noise is purely a theoretical and abstract substance - in fact we know from [39] that Brownian motion is almost never differentiable. Therefore it is not a typical function, in the way we consider standard functions in real analysis. For the purposes of the computational part of this project, we consider white noise to obey certain properties - the same as used in applied mathematics literature. We start with the following:

$$\xi(\cdot) \ast f(\cdot) := \int_0^t f(t - s)\, dB_s \quad (7)$$

which comes from the mathematician’s definition of white noise. Since we know from [39], if $f(\cdot) \in L^2(\mathbb{R})$ that equation-7 will be well defined with

$$\mathbb{E}[(\int_0^t f(t - s)\, dB_s)^2] = \int_0^t f(t - s)^2\, ds \quad (8)$$

this is Ito’s isometry in e.g [33], suggesting that the definition in 7 also satisfies (part) of the engineer’s definition of white noise. We also want to be able to manipulate white noise somewhat with respect to the Laplace transform, with notation $L$ or $\tilde{\cdot}$.

Let

$$|L(\xi(\cdot))(s)|^2 = 1 \quad (9)$$

which comes from the musician’s definition. Related to this operation, is the idea that we want to “generalise” the Laplace transform to white noise, for further use. We require

$$L(f(\cdot) \ast \xi(\cdot)) = L(f(\cdot)) \times L(\xi(\cdot)), \quad L^{-1}(L(\xi(\cdot))) = \xi, \quad \xi_1 \perp \xi_2 \implies L(\xi_1) \perp L(\xi_2)$$

$$\mathbb{E}[L(\xi_k)] = 0 \quad (10)$$

All in all, we want to be able to treat white noise as some variant of normal functions. There is a method of rigorously defining white noise from mathematical principles, by considering them as generalised functions on an infinite dimensional space [26]. This is a very complicated and theoretical mathematical definition, which seems to be universally ignored by all applied mathematicians, physicists and engineers, that use this concept in modelling.

In this thesis, we often find it useful to look for fixed points of our de-randomised system. This could be useful as we might want to consider the dynamics of the model to gravitate towards certain points due to its non-random deterministic part and then to fluctuate around them due to the white noise input. To perform this analysis, we would temporary remove the randomness input from the model - i.e take $\delta \xi = 0$ and we
would then analyse the remaining deterministic system as in [18],[40]. This approach can be useful, as a system having an attractive fixed point characterises healthy brain activity, whilst brain-wide limit cycles are associated with problematic and unhealthy states, as the example in figure-1 showed.

1.2.3 The Synaptic Response Function

We have described in sections-1.2.1,1.2.2 the types of input that a neural mass may receive. We thus now need to discuss exactly how these inputs will alter the voltage of a specific neural mass. A way of thinking about how this could be modelled, is by considering this as an input-output black-box system, where both inputs and outputs are functions dependent on time. We present a general overview of this concept and how we model it, in appendix-A. Reusing notation from that description, we present a widely used impulse function for modelling the synaptic response function:

$$T(u(\cdot)) = h(\cdot) * u(\cdot)(t), \quad h(t) := \frac{\alpha\beta}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})I_{[0,\infty)}(t) \quad \beta > \alpha > 0 \quad (11)$$

where $I$ is the indicator function and $\ast$ refers to a convolution. This function describes how an incoming signal (which in reality is an action potential) perturbs the average voltage of the post synaptic population - thus it models the dynamics of the synapses on a macroscopic level. $\alpha, \beta$ refer to the decay and rate properties of the synapse respectively. As it is useful to normalize $h$ so that $\|h\|_{L^1} = 1$, we model the strength of synapses (also the the number of incoming axons) by scaling $h$ with parameters to be introduced. For this thesis, we take $\alpha = 50, \beta = 200$ and therefore $h$ will look like:

![Figure 3: A plot of the impulse response function $h(t)$ from equation-11, with $\alpha = 50, \beta = 200$](image)

the significance of the spike at time $r$ ($\approx 0.01s$ in the figure) of the synaptic response function, is that for fixed time $t$, the input that a neural mass receives at time $t - r$ will contribute more to its present voltage than inputs arriving at different times in the past.

We also note the important fact, that $h$ satisfies the following second order O.D.E:

$$h''(t) + (\alpha + \beta)h'(t) + \alpha\beta h(t) = 0 \quad (12)$$
this causes the second order S.D.E that we obtain in equations-27,50. Taking the Laplace transform of 12 now presents us

\[(s^2 + (\alpha + \beta)s + \alpha\beta)\tilde{h}(s) = \alpha\beta\]  

which will be exploited later on.

To make the model more realistic, we also multiply these inputs with factors to signify the ability of the synapses being able to process these inputs. The values of these factors may vary for different neural masses and even for different types of inputs for the same neural mass. A simple example of this, is the additional voltage caused for a neural mass \(V_x\) from another neural mass \(V_y\):

\[\mu_{y,x} \times h(\cdot) \star S(V_y(\cdot)) \]  

Here \(\mu_{y,x}\) corresponds to the strength of the synapse (number of incoming axons) for the input of neural mass \(x\) from neural mass \(y\). If \(\mu_{y,x} > 0\) then we say that the neural mass \(y\) is excitatory for neural mass \(x\) i.e the presence of a voltage in \(y\) will increase the voltage in \(x\). On the other hand, taking \(\mu_{y,x} < 0\) means the neural mass \(y\) is inhibitory for neural mass \(x\) i.e. the presence of a voltage in \(y\) will reduce the voltage in \(x\). A value of \(\mu_{y,x}\) of a large magnitude, would suggest that the signals from neural mass \(y\) to neural mass \(x\) would influence its behaviour somewhat.

Now we look at the voltage in \(x\) due to external inputs:

\[v_x \times h(\cdot) \star Q_{in}(\cdot) \]  

here the parameter \(v_x \geq 0\) defines the mean synaptic efficiency of a neural mass. A higher value will indicate that the neural mass is more susceptible to being influenced by signals sent to it from external sources, as it is somehow better at processing these types of signals.

1.2.4 The Standard Parameters

We note that the parameters that we introduced in this subsection \(v, \theta, Q_{max}, \delta, \alpha, \beta\) correspond to real biological parameters and thus have units. One can find their values using experiments, for the rest of this section we fix the parameters (unless specifically stated) as:

\[v = 0.001\]
\[\theta = 0.015\]
\[Q_{max} = 250\]

for parameters in section-1.2.1,
\[\delta = 0.1\]

for the variance of white noise section-1.2.2,
\[\alpha = 50\]
\[\beta = 200\]

for parameters described in section-12.

We call the values of fixed parameters above the standard parameters.

Note that we do not include the values of \(\mu, q\) in the above as they are free parameters and we will be modifying these in this section to see if they change our results.
1.3 A Simple Neural Mass Model

Now that we have described the general parameters needed, as an example we present pretty much the simplest version of a neural mass model that we could consider:

\[ V(t) = h(\cdot) \ast v\xi(\cdot) \quad V(0) = 0 \]  

(16)

the input here is only white noise \((Q_m = \xi)\). We visualize this with:

\[ \xi(s) \xrightarrow{vh(\cdot)} V(t) \]

Figure 4: A visual representation of equation-16

For this subsection, we present a novel value of \(h\). This is to further simplify the model and to show the effect of changing \(h\) on future models. Only within this subsection we define:

\[ h(t) := \alpha e^{-\alpha t} 1_{[0,\infty)}(t) \]  

(17)

Note that this is approximately the same as equation-11 when \(\beta >> \alpha\), when we ignore the rise-rate part and the synaptic response function is assumed to act instantaneously. We now want to be able to write 16 as a S.D.E. Taking the naive Laplace transform

\[ \tilde{V}' + \alpha \tilde{V} = s\tilde{V} - V(0) + \alpha \tilde{V} = \tilde{V}(s + \alpha) = v\bar{h}(s)Q_m(s + \alpha) = v\alpha \tilde{Q}_m \]  

(18)

and taking the inverse naive Laplace transform produces

\[ V' + \alpha V = v\alpha Q_m \]  

(19)

where the existence of equation-19 is due to the relation \(h' + \alpha h = 0\). If we had used \(h\) from equation-12, we would have obtained a second order S.D.E.

Indeed our system from equation-16 is so simple, that we can in fact find a complete theoretical expression for it. Combining equations-7,16 produces:

\[ V(t) = v \int_0^t h(t - u) \, dB_u = v\alpha e^{-\alpha t} \int_0^t e^{\alpha u} \, dB_u \]  

(20)

Note that the stochastic integral in the above makes sense as the \(t\) is outside. By appendix-B we can express

\[ \int_0^t e^{\alpha u} \, dB_u = B_{\frac{1}{2}}(e^{2\alpha t} - 1) \]  

(21)

and therefore by the scaling property of Brownian motion we can then combine equations-23,21 and obtain

\[ V(t) \overset{D}{=} B_{\frac{1}{2}}(1 - e^{-2\alpha t}) \]  

(22)

We want to test to see whether simulations obtained from equation-19 with appendix-C correspond to the theoretical derivation in equation-16 (they should!). One way of comparing to see if they have the same distribution, is a QQplot:
Figure 5: A QQ Plot of 500 runs of equation-19 with appendix-C where $T = 5$, against samples from the distribution of $V(5)$ from equation-23. $\alpha = v = 5$

Figure-5 indeed suggests that our theoretical result agrees with our simulated results. Our methods, although questionable mathematically, seem to both produce correct results and to be consistent.

Now examining the power spectrum of our voltage see appendix-D, we obtain

$$F_V(\omega) = |\tilde{V}(\cdot)(i\omega)|^2 = \frac{v^2}{\omega^2 + \alpha^2}$$

where we used equation-9. With respect to our system figure-4, we have an input white noise which has equal power at all frequencies, whilst the output has lower power for higher frequencies i.e $\omega \rightarrow F_V(\omega)$ is decreasing. We thus describe the effect of the “black box” in figure-4 to be a low pass filter. However this scenario is not realistic at all, as real EEG power spectrums often have peaks e.g figure-21a.

We were “fortunate” that we were able to derive the distribution of our simple system. We will not make this a habit, as more complicated models will not be so conveniently analysable.

1.4 A Model With Feedback

Now we consider a more complex model. We include a feedback loop from the mass to itself so it will be self-referential:

$$V(t) = vh(\cdot) \star Q_{in}(\cdot) + \mu h(\cdot) \star S(V(\cdot)), \quad V(0) = 0$$

easily visualised as:

![Diagram of equation-24]

Figure 6: A visual description of equation-24

We want to write equation-24 as an S.D.E. The trick is to take the naive Laplace transform of it to obtain

$$\tilde{V}(s) = \tilde{h}(s)(v\tilde{Q}_{in} + \mu \tilde{S}(V(\cdot)))$$
now we use equation-13 to produce

\[(s^2 + (\alpha + \beta)s + \alpha \beta)\tilde{V}(s) = \alpha \beta (v\tilde{Q}_m + \mu \tilde{S}(\cdot))\]  \quad (26)

taking the reverse naive Laplace transform and also assuming that \(V''(0) = 0\), we obtain our S.D.E

\[V''(t) + (\alpha + \beta)V'(t) + \alpha \beta V(t) = \alpha \beta vQ_m(t) + \alpha \beta \mu S(V(t))\]  \quad (27)

Remember that the order of this S.D.E was satisfied by the order of the solution of equation-26, which directly came from the second-order solution of \(h\) in equation-12. If we had used the \(h\) in equation-17, then equation-27 would have been an S.D.E of order 1.

Now the fixed points of the deterministic variant of equation-27 satisfy the relation

\[V^* = vq + \mu S(V^*)\]  \quad (28)

Unfortunately \(S\) is non-linear and so we cannot derive a closed expression form for the solution of \(V^*\). But we can see that the number of fixed points that we have depends on our values of \(\mu, v, q\) that we take. Plotting the two functions \(S(V), g(V) = -vq/\mu + V/\mu\) together, the x-axis points of the intersections are the fixed points. We provide two examples:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Two plots with standard parameters varying the values of \(\mu, q\). The x-axis values of the intersections satisfy equation-28.}
\end{figure}

a) \(\mu = -0.01, q = 500\), fixed points (0.010760) (6dp)

b) \(\mu = 7.5 \times 10^{-5}, q = 5\) fixed points (0.0058470, 0.016150, 0.022182) (6dp)

In figure 7 we see that we can have different numbers of fixed points depending on our parameters - these can be found by our optimisation algorithm (see appendix-C). Indeed with this approach we can actually characterize exactly how many fixed points we will have on the \((q, \mu)\) plane:
Figure 8: The number of fixed points that we have with the standard parameters, varying \((q, \mu)\). The interior of the black region corresponds to the system having three fixed points, the interior of the white region corresponds to the system having a single fixed point and the boundary between the two is when the system has a bifurcation.

We immediately note that for a large fixed input (large \(q\)) the system will be unstable and have multiple fixed points - it is overloaded in some manner. We will now see that the stability properties of the fixed points depend on how many of them there are.

1.4.1 Bifurcations

From figure-8 we see that for some fixed values of \(q\), varying \(\mu\) will lead to a different number of fixed points. We can visualize this in the same fashion as figure-7:
Figure 9: Three plots with standard parameters, $q = 5$ where
a) $\mu = 5e - 5$  b) $\mu = 7e - 5$  c) $\mu = 7.5e - 5$.
The x-axis values of the intersections satisfy equation-28

To calculate the stability of fixed points we write equation-27 as

$$V'(t) = W(t), \quad W'(t) = \alpha \beta v q + \alpha \beta \mu S(V(t)) - (\alpha + \beta)W(t) - \alpha \beta V(t)$$ (29)

and the Jacobian of this system is

$$J_{V,W}(V,W) = \begin{pmatrix} 0 & 1 \\ \alpha \beta \mu S'(V(t)) - \alpha \beta & -\alpha - \beta \end{pmatrix}$$ (30)

the eigenvalues of the above at $(V^*, W^*)$ are

$$\lambda_1, \lambda_2 = \frac{-(\alpha + \beta) \pm \sqrt{(\alpha + \beta)^2 + 4G}}{2}, \quad G := \alpha \beta \mu S'(V^*) - \alpha \beta$$ (31)

we can conclude

$$Re(\lambda_2) < 0, \quad Re(\lambda_1) = \begin{cases} > 0 & \text{if } G > 0 \\ 0 & \text{if } G = 0 \\ < 0 & \text{if } G < 0 \end{cases}$$ (32)

With the parameters that we use in figure-9, we use numerical methods to see how many fixed points we have, along with their values when plugged into equation-32. We obtain three different scenarios depending on $\mu$:

Figure 10: Three plots, each on the $(V, W)$ axis of figure-9, corresponding respectively to
1) $\mu < 7.1 \times 10^{-5}$ a single stable (low) fixed point e.g figure-9a.
2) $\mu = 7.1 \times 10^{-5}$ (6dp) we have a lower stable fixed point and an upper saddle node - indeed we have a saddle node bifurcation e.g figure-9b.
3) $\mu > 7.1 \times 10^{-5}$ the lower fixed point is stable and we have two upper nodes, the middle is unstable and the highest is stable e.g figure-9c.

We progress from left to right by increasing $\mu$ - causing a saddle fixed point to appear out of nowhere, which then immediately splits into a an unstable and stable fixed point which drift apart.
As we increase $\mu$, the lower stable node and the middle unstable fixed point on the right of figure-10 will continue to drift together. They will end up colliding and producing a (lower) bifurcation at $\mu = 1.36 \times 10^{-4}(6dp)$. Effectively the events in figure 3 will run backwards with the fixed points “reflected”. We can see this double bifurcation unfold in the following bifurcation diagram:

![Bifurcation Diagram](image)

Figure 11: A bifurcation diagram with standard parameters, $q = 5$, $\mu$ is the bifurcation parameter

These double bifurcations only occur when we have self-excitatory neuronal populations $\mu > 0$ (recall equation-14). In the case that the neuronal populations are self-inhibitory ($\mu < 0$) then the gradient of $S(V)$ will be of the opposite sign to that of $g(V)$ and we will always have a solitary fixed point for any $q$. From equations-30,32 we see that at the Jacobian at the fixed point has negative eigenvalues with negative trace, implying that the solitary fixed point would be stable.

### 1.4.2 Bi-stability

For certain values of parameters, our equation-24 will have two fixed points, e.g for the standard parameters $q = 5, 7.1 \times 10^{-5} < \mu < 1.36 \times 10^{-4}$ like in figure-11. Since our system has a random input, the voltage of the system may fluctuate around a fixed point, then after some time the random noise input may be big enough to move it to gravitate around the other fixed point. To show an example of this phenomenon, we plot a sample time series of equation-24, with the same parameters as in 7-b below:
The voltage often fluctuates between the fixed points. We want to examine the contribution of the zero-mean white noise to the system. As at the fixed point, the voltage would remain the same without this random addition, we want to calculate the value of:

\[ X_t := v h(\cdot) \ast Q_{in}(\cdot)(t) =: \delta v \int_0^t h(t - s) \, dB_s \]  \hspace{1cm} (33)

using our definition of \( h(\cdot) \) from equation-11 we obtain

\[ X_t = \frac{\delta v \alpha \beta}{\beta - \alpha} [e^{-\alpha t} \int_0^t e^{\alpha s} \, dB_s - e^{-\beta t} \int_0^t e^{\beta s} \, dB_s] \]  \hspace{1cm} (34)

as equation-34 is well-defined by our definition of the stochastic integral, then equation-33 will also be well defined and we can can consider the stochastic process \( X_t \) pathwise. By Ito’s isometry we have

\[ < \int_0^t e^{\gamma s} \, dB_s > = \int_0^t e^{2\gamma s} \, ds = \frac{1}{2\gamma}(e^{2\gamma t} - 1), \ \gamma \in \mathbb{R} \]  \hspace{1cm} (35)

by the result in appendix-B we combine 34, 35 to obtain up to distribution

\[ X_t = \frac{\delta v \alpha \beta}{\beta - \alpha} [e^{-\alpha t} B_{\frac{1}{2\alpha}}(e^{2\alpha t} - 1) - e^{-\beta t} B_{\frac{1}{2\beta}}(e^{2\beta t} - 1)] \]  \hspace{1cm} (36)

by the scaling property of Brownian motion we obtain

\[ X_t = \frac{\delta v \alpha \beta}{\beta - \alpha} [B_{\frac{1}{2\alpha}}(1 - e^{-2\alpha t}) - B_{\frac{1}{2\beta}}(1 - e^{-2\beta t})] \]  \hspace{1cm} (37)

recalling that \( \beta > \alpha \) we notice the following properties of the following function

\[ f(t) := \frac{1}{2\alpha}(1 - e^{-2\alpha t}) - \frac{1}{2\beta}(1 - e^{-2\beta t}), \ \ \ f(0) = f'(0) = 0, f'(t) > 0 \ \ t > 0 \]  \hspace{1cm} (38)

which implies

\[ f(t) \geq 0 \ \ t \geq 0, \ \ f : [0, \infty) \to \mathbb{R} \ \text{is bijective} \]  \hspace{1cm} (39)
combining equations-37,39 use the stationary property of Brownian motion, along with scaling to obtain

\[ X_t \sim B_{g(t)}, \quad g(t) := \left( \frac{\delta \nu \alpha \beta}{\beta - \alpha} \right)^2 \left( \frac{1}{2\alpha} (1 - e^{-2\alpha t}) - \frac{1}{2\beta} (1 - e^{-2\beta t}) \right). \] (40)

It is very convenient that we can express this stochastic integral as B.M as we know that it will have many nice properties - more specifically is is space invariant, fulfils the reflection principle and fulfils the strong Markov property. These properties can try and help us see how often equation-33 switches between level sets:

By equation-39 we obtain

\[ P \left[ \sup_{0 \leq t \leq T} X_t \geq a \right] = P \left[ \sup_{0 \leq u \leq g^{-1}(t) \leq T} B_u \geq a \right] = P \left[ \sup_{0 \leq u \leq g(T)} B_u \geq a \right] \quad a > 0 \] (41)

by the reflection principle of B.M we obtain

\[ P \left[ \sup_{0 \leq t \leq T} X_t \geq a \right] = 2 P \left[ B_{g(T)} \geq a \right] \quad a > 0 \] (42)

we want to find the distribution of the stopping time

\[ R_a := \inf \{ t > 0, B_{g(t)} \geq a \} \] (43)

from equation-42 we now obtain

\[ P[R_a \leq t] = 2 P[N(0, g(t)) \geq a] = 2 \int_a^{\infty} \frac{e^{-\frac{x^2}{2g(t)}}}{\sqrt{2\pi g(t)}} \, dx = 2 \int_a^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \, dy \] (44)

differentiating the above will present us the density of \( R_a \):

\[ f_{R_a}(t) = \frac{\delta}{\delta t} P[T_a \leq t] = \frac{2ag'(t) e^{-\frac{a^2}{2g(t)}}}{(\sqrt{g(t)})^3 \sqrt{2\pi}} \] (45)

Unfortunately the expectation of \( R_a \) will be infinite - we use a nice result to see this. Recall from [38] that

\[ T_a := \inf \{ t > 0 | B_t \geq a > 0 \}, \quad \mathbb{E}[T_a] = \infty \] (46)

we define \( h(t) := t - g(t) \) and noting that \( f'(t) \to 0 \) we have for small \( C \) that \( h'(t) = 1 - C^2 f'(t) \geq 0 \) \( t \geq 0 \) which produces the results

\[ t \geq g(t) \quad i.e \ g^{-1}(t) \geq t \] (47)

as \( g(R_a) = T_a \) we use equations-46,47 and conclude

\[ \mathbb{E}[R_a] = \mathbb{E}[g^{-1}(T_a)] \geq \mathbb{E}[T_a] = \infty \] (48)

If a system was only to expressed by \( X_t \) we could find how often it would switch between the fixed points using equation-44. However the system that we are analysing equations-24,27 has other deterministic terms. It might then be approximable when both \( \mu, q \) are very small. Whilst this does seem to be the case, rounding errors of having values very close to zero, as well as the still-present effect of the deterministic terms, makes these theoretical derivations less effective and further away from simulated results than the author would have desired.
1.5 Coupled Neural Masses

We now consider a more complex and realistic model, which incorporates the transmission of signals between two neural masses. The introduction of an interaction between neural masses is particularly important, as the model we now introduce can display periodic rhythms which can describe the types of unhealthy states in subsection-1.1.

We have two neural masses, one excitatory for the other - denoted $V_e$ and the other inhibitory for the other - denoted $V_i$ (see equation-14). This is a more realistic model as local cortical circuits are comprised of both excitatory and inhibitory neurons. We could write such a model as:

$$V_e(t) = v h(\cdot) * Q_{in}(\cdot) + \mu_{ie} h(\cdot) * S(V_i(\cdot))$$

$$V_i(t) = \mu_{ei} h(\cdot) * S(V_e(\cdot))$$

(49)

where $\mu_{ei} > 0, \mu_{ie} < 0$ by definition of the inhibitory/excitatory masses. A visual representation of this model is:

![Figure 13: A visual description of equation-49](image)

With almost the same calculation that produced equation-27, we can obtain a double second order S.D.E:

$$V''_e(t) + (\alpha + \beta)V'_e(t) + \alpha \beta V_e(t) = \alpha \beta v q + \alpha \beta v \delta \xi(t) + \alpha \beta \mu_{ie} S(V_i(t))$$

$$V''_i(t) + (\alpha + \beta)V'_i(t) + \alpha \beta V_i(t) = \alpha \beta \mu_{ei} S(V_e(t))$$

(50)

Again we want to recover the fixed points from the related deterministic system (set $Q_{in} = q$) and with this we obtain

$$V^*_e = v q + \mu_{ie} S(V^*_i)$$

$$V^*_i = \mu_{ei} S(V^*_e)$$

(51)

from these we obtain a closed form solution for both fixed points:

$$0 = -V^*_e + v q + \mu_{ie} S(\mu_{ei} S(V^*_e))$$

$$0 = -V^*_i + \mu_{ei} S(v q + \mu_{ie} S(V^*_i))$$

(52)

we can use optimisation algorithms to find approximate values of $(V^*_e, V^*_i)$.

We would also like to analyse the stability of these fixed points theoretically. One way could be to calculate the Jacobian - in the same way as in equation-30, unfortunately this would result in having to deal with a $4 \times 4$ matrix. A simpler method is to use a linear approximation and then use the method in appendix-A where we consider the fluctuations of $(V_e(t), V_i(t))$ about their respective fixed points. Subsequently we can use simulated results to see if they agree with our theoretical derivations. Firstly we want to deal with the non-linearity of $S(\cdot)$ - luckily we can use a trick since we are considering the case that $(V_e(t), V_i(t))$ are close to $(V^*_e, V^*_i)$. This involves Taylor’s theorem which in this case produces:

$$S(V_k(t)) = S(V^*_k(t)) + S'(V^*_k(t))(V_k(t) - V^*_k), \quad k = e, i$$

(53)
we can easily substitute this into the deterministic variant of equation-49 to obtain the following equations in terms of the fluctuations:

\[
\begin{align*}
\overline{V}_e(t) &= h(\cdot) \ast (v\delta(\cdot) + G_{ie}\overline{V}_i(\cdot)) \\
\overline{V}_i(t) &= h(\cdot) \ast G_{ei}\overline{V}_e(\cdot) \\
\overline{V}_k(t) &= V_k(t) - V_k^*, \quad k = e, i \\
G_{ie} &= \mu_{ie}S'(V_i^*), \quad G_{ei} = \mu_{ei}S'(V_e^*)
\end{align*}
\] (54)

we use this to determine the transfer functions of the random input to the voltage, they are:

\[
\begin{align*}
H_{Q_0V_e} &= \frac{\tilde{v}h(s)}{1 - G_{eie}\tilde{h}^2(s)}, \quad H_{Q_0V_i} = \frac{\tilde{v}h^2(s)G_{ei}}{1 - G_{eie}\tilde{h}^2(s)}, \quad G_{eie} = G_{ei}G_{ie}
\end{align*}
\] (55)

We note the poles of both of the transfer functions are the same. So we want to find the solution of

\[
1 - G_{eie}\tilde{h}(\lambda) = 0 \iff (\lambda + \alpha)^2(\lambda + \beta)^2 = \pm G_{eie}(\alpha\beta)^2
\] (56)

noting that both sides of the above are real, we take the square root and arrange it into a quadratic function

\[
\lambda^2 + (\alpha + \beta)\lambda + \alpha\beta \pm i\alpha\beta\sqrt{-G_{eie}} = 0
\] (57)

where we know \( G_{eie} < 0 \) from equation-54. We thus obtain four solutions for the resonances of equation-55:

\[
\lambda_{1,2,3,4} = \frac{-\alpha - \beta \pm \sqrt{(\alpha + \beta)^2 - 4\alpha\beta \pm 4\alpha\beta\sqrt{-G_{eie}}}}{2}
\] (58)

taking the square root of the complex number, we can easily deduce

\[
\max\{R(\lambda_1), R(\lambda_2), R(\lambda_3), R(\lambda_4)\} < 0 \iff G_{eie} > \frac{(\alpha + \beta)^2}{\alpha\beta} =: G_{\text{crit}}
\] (59)

and a Hopf bifurcation (a transition between stability and instability) should take place when \( G = G_{\text{crit}} \). We soon see that it is a supercritical Hopf bifurcation (see appendix-B).

Again we now want to compare our theoretical result obtained with simulations, to see if our theoretical approach we took was correct. Fixing \( \mu_{ei} = -\mu_{ie} = 1 \) with our standard parameters, we see \( G_{\text{crit}} = 6.25 \) and that \( G_{eie} \) crosses this value in the interval \( q \in [1659, 1660] \). We examine some simulated time series:
Figure 14: Four different time-series simulations of $V_e(t)$ (combining equation-50 with appendix-C) with standard parameters, $\mu_{ei} = -\mu_{ie} = 1$ with:

a) $q = 1655$  
b) $q = 1659$  
c) $q = 1660$  
d) $q = 1665$

we have also draw in the either the unique, or middle fixed point.

It seems that our simulated results agree with our theoretical derivation. There is a clear stochastic limit cycle in figure-14d that is not in figure-14a and this is because of a supercritical Hopf bifurcation that happens between the two - see appendix-B. Actually we predict that this bifurcation happens between figures-14b,14c, but we cannot see this clearly from the plots as the derived limit cycle would be small just after the bifurcation, see appendix-B.

Now we want to see whether we can obtain bifurcations if we fix $q$ and vary $\mu_{ei}, \mu_{ie}$. So we fix a value of $q$ where we know that there is no limit cycle for some values of $\mu_{ei}, \mu_{ie}$, e.g $q = 200$. Now we plot to see what values of $|G_{eie}|$ we can obtain:
Figure 15: Plot $|G_{ei}|$ (Z-axis) against $\mu_{ei} \geq 0, \mu_{ie} \leq 0$ (X-Y axis), with standard parameters and $q = 200$.

Another way of visualising this is with two-dimensional plots. We provide an example where we perform this along the $(a, -a)$ - line:

Figure 16: Plot $|G_{ei}|$ against $|\mu_{ei}| = \mu_{ei} = -\mu_{ie}$. Also add the value of $|G_{crit}|$. We use standard parameters $q = 200$.

In this scenario we will have a limit cycle when $|\mu_{ei}| = \mu_{ei} = -\mu_{ie} \geq 0.121 (3dp)$. In a different scenario to figure-14, increasing $|\mu_{ei}|$ past a point whilst keeping $q$ fixed does not produce bifurcations. Looking at figure-15 we see that the same is not true for all limits on the $(-\mu_{ie}, \mu_{ei})$ plane.

We now want to see the “limiting” behaviour of $\mu_{ei}$. We now see that if we take $\mu_{ie} = -0.0035, \mu_{ei}$ large we still get limit cycles:
Figure 17: Standard parameters, \( q = 200 \)

a) A plot with a larger range to compare with figure-15

b) A time series plot where we have a limit cycle. \( \mu_{ie} = -0.0035, \mu_{ei} = 100.0 \).

To prevent limit cycles, it seems like a good idea to decrease the negative value of \( \mu_{ie} \), i.e., we want the inhibitory neural mass to have a greater effect in reducing the voltage of the excitatory neuron. This seems to prevent some transitions to limit cycles, if other parameters are varied. A potential explanation is that this operation is similar to the idea of having a smaller fixed input \( q \), as the excitatory neuron would be “less overloaded” in some fashion.

This type of analysis, where we vary coupling parameters between neural masses and see if a bifurcation can be possible, can be enlightening. Indeed in a more complex/realistic model e.g equation-63, modifying the corresponding variant of these coupling parameters produces the same effect. This can have a biochemical meaning as well, as changing these coupling parameters can relate to the taking of certain drugs, see for example [12]. People suffering from problems such as epilepsy can then be recommended to take these drugs, so that they do not suffer their own unpleasant version of these limit cycles.
2 A Model for Pan-Cortical Activity

2.1 Introduction

In section-1 we discussed how local neural masses on the cortex may be modelled, as well as examples of behaviours produced by limited interactions between them. This was of course an obvious prelude to a much bigger goal: attempting to create a model that can predict cortical activity on the macroscopic scale. Our aim in this section is more specific: we are interested in investigating which regions generate the so-called alpha rhythm in human resting state (when one is not conducting strenuous mental activities). This rhythm has a prominent frequency present in brain voltages, which lies between 7 and 13 Hz, but its precise value is slightly different for each person. Much further biological and physical properties of this rhythm are described in [31]. Of course there are many models that one could consider though most researchers use different combinations of concepts that were considered in section-1. Such models include early efforts in [37],[44], as well as more recent developments in [19],[22],[28].

Having fixed the general form of the model which uses certain parameters, we want to find specific values of these parameters (also called a dynamical working point) that allows us to obtain more realistic results for modelling how alpha rhythms propagate through the cortex. Whilst some parameters (or the relation between these parameters) can be obtained via experimental observations, others, specifically the coupling strength between different brain regions, cannot be determined in such a fashion. Thus we consider these to be free parameters and we attempt to find their values via an inverse problem, with real data to help us.

Being cognisant of this, a method of finding a dynamical working point is by comparing specific metrics gathered from both experimental results and from simulations of the model. Varying free parameters in the model will change the value of this metric, allowing us to determine our parameter set in the model that produces a metric that is close to the one observed experimentally. This is a typical attempt to solve an inverse problem, a method that is pervasive in computational neuroscience. There are of course many different metrics that we may consider - we want to of course choose a “good metric”, but what this is could be subjective. We are fully aware that different metrics will simply correspond to different types of “similarities” between the model and the experimental results.

In this section we thus describe a neural-mass model of cortical dynamics. We then present experimental data to be used: time series gathered by MEG and a structural connectivity matrix gathered by diffusion tensor imaging based on probabilistic tractography (for calculating real-life brain connections). These things combined, we attempt to fit the model to obtain values of our final free parameters. With these same resources, we will then attempt to estimate a specific parameter of the model.

2.2 A Neural Mass Model

We recall the reader to the functions and concepts presented in subsection-1.1, as we will use a variation of the concepts presented there for our new model. We now present a variant of the model from [14] in this subsection, a neural mass model known as the Jansen-Ritt model [20],[24].

Before presenting the formulation, we note some major differences with what we have encountered in section-1. The first difference is that we now consider each local region
to consist of three different types of neurons - excitatory, inhibitory and pyramidal. We have already examined the first two types in subsection-1.5. Pyramidal neurons have the property of being connected to both other types of neurons in their local region, as well as being the only types of neurons in each region that are connected to other regions. Furthermore they are the dominant type of neuron for producing the voltage potential within the setting of a single local region, so we consider the local voltage potential to be that of the pyramidal neuron only. For more biological information on these types of neurons see [25]. Furthermore we observe that not only are we considering a local model, but that there will be some sort of connection between this local model and other local models. Every inter-regional connection will be scaled by a global coupling parameter $K$.

Now we present the model used in this section:

$$V^k_p(t) = h_e(\cdot) \cdot (q + \sigma^k \xi^k(\cdot) + \gamma_2 S(V^k_p(\cdot))) + K \sum_{l \neq k} \beta_{l,k} S(V^l_p(\cdot)) - h_i(\cdot) \cdot \gamma_4 S(V^k_i(\cdot))$$

$$V^k_i(t) = h_e(\cdot) \cdot \gamma_3 S(V^k_e(\cdot)) - h_i(\cdot) \cdot \gamma_5 S(V^k_i(\cdot))$$

$$V^k_e(t) = h_e(\cdot) \cdot \gamma_1 S(V^k_p(\cdot))$$

$k = 1, \ldots, N, \gamma_{1,2,3,4} > 0, \xi^k \perp \perp \xi^l \neq l, q, \sigma \geq 0$

if any of the parameters presented look similar to those described in subsection-1.1, it is because they are. The reader should be familiar with the biological interpretations of the constants and functions $V, S, q, \xi, h$. The “variance” of the white noise is now denoted by $\sigma$ instead of $\delta$. Extra modifications include a new definition of $S(\cdot)$ (originally equation-5):

$$S(V) := \frac{1}{1 + e^{-\rho_1(V - \rho_2)}} \quad \rho_1, \rho_2 > 0$$

Additionally we refine the definition of $h$ (originally equation-11) to the following functions:

$$h_x(\cdot) := H_x \kappa_x t e^{-t \kappa_x}, \kappa_x, H_x > 0 \quad x = e,i$$

where $H_x, \kappa_x$ refers to the synaptic efficiency and rate-constant respectively. We have refined the model to include different types of $h$ to discern the difference between different synaptic responses for different neural mass types. Additionally, the initial condition of the system will be the fixed points of its deterministic variant - these will exist and be stable in the situations that we examine (discussed in subsection-2.3). As we eventually need to consider the system to be fluctuating around mean-0 (we want to compare it to data that has this property) we also need to consider the variant of equation-63:

$$V^k_x(t) := V^k_x(t) - \bar{V}^k_x \quad x = p, i, e, k = 1, \ldots, N$$

which is just a translation. In any case, we are not so much interested in the values of the fixed points as in the stochastic fluctuations around them. Here is a nice picture to help one visualise our new system:
2.3 Manipulating the Model

After our model description, we want to find out some of its theoretical mathematical properties, so that we can use mathematical techniques to try and describe the model (and thus hopefully real-life) activity. Eventually we will use these descriptions and compare them against real experimental data. This is analogous to operations in section-1. Firstly we describe how to calculate the Jacobian of equation-63. Additionally we use a linear approximation to calculate the theoretical cross-power spectrum and covariance/correlation matrix of equation-63. Performing a linear approximation in some form with a neural mass model, is a common technique and is used in [6], [10], [14], [28].

Whilst we could perform simulations of equation-63 by using appendix-C, we present an example in figure-20 to show how this process may be very computationally costly.

Thus we present these derivations which were described in a past version of [14]:

Firstly we note that in exactly the same way that we derived S.D.E’s for all of the neural mass models considered in section-1, we rewrite equation-63 as a system of
$10 \times N$ coupled non-linear S.D.E’s:

\[
\begin{align*}
V \quad k &= 1, \ldots, N \\
\dot{V}^k_p(t) &= U^k_p(t) \\
\dot{U}^k_{p,1}(t) &= -2\kappa_e U^k_{p,1} - \kappa_e^2 V^k_{p,1} + \kappa_e H_e (q + \sigma^k \xi(t)) + \gamma_2 S(V^k_e(t)) + K \sum_{l \neq k} \beta_{l,k} S(V^l_p(t)) \\
\dot{V}^k_{p,2}(t) &= U^k_{p,2}(t) \\
\dot{U}^k_{i,1}(t) &= -2\kappa_i U^k_{i,1} - \kappa_i^2 V^k_{i,1} + \kappa_i H_i \gamma_4 S(V^k_i(t)) \\
\dot{V}^k_{i,2}(t) &= U^k_{i,2}(t) \\
\dot{U}^k_e(t) &= -2\kappa_e U^k_e - \kappa_e^2 V^k_e + \kappa_e H_e \gamma_1 S(V^k_e(t)) \\
V_p &= V_{p,1} - V_{p,2} \quad V_i = V_{i,1} - V_{i,2}
\end{align*}
\]

(64)

Using appendix-C we can conduct simulations of our model and we can also compute the linear stability of our model (like in section-1 see appendix-B), from the eigenvalues derived from the system above.

To find the fixed points (denoted by $\gamma$) of the deterministic variant of equation-66, means solving the set of equations:

\[
\begin{align*}
\overline{V}^k_p &= \epsilon_e (q + \gamma_2 S(\overline{V}^k_e) + K \sum_{l \neq k} \beta_{l,k} S(\overline{V}^l_p)) - \epsilon_i \gamma_4 S(\overline{V}^k_i) \\
\overline{V}^k_i &= \epsilon_e \gamma_3 S(\overline{V}^k_p) - \epsilon_i \gamma_5 S(\overline{V}^k_i) \\
\overline{V}^k_e &= \epsilon_e \gamma_1 S(\overline{V}^k_p) \\
&\quad k = 1, \ldots, N \quad \epsilon_x = \frac{H_x}{\kappa_x} \quad x = i, e
\end{align*}
\]

(65)

Fortunately these equations can be solved numerically.

Now to find the Fourier spectrum, we use a linear approximation (mostly of the non-linear $S(\cdot)$) around the fixed points $(\overline{V}^k_p, \overline{V}^k_i, \overline{V}^k_e)$, $k = 1, \ldots, N$, which is the Taylor series up to the first order term. One could also use higher order terms, but it considerably complicates the calculations needed to be performed by hand. We know that this will be a “good” approximation when we will be close to the fixed point i.e when $V - \overline{V}$ is close in some manner. Whether this approximation works or not will be discussed later.

Starting with equation-63, we take the naive Laplace transform of it (which has
Additionally define the block matrix $M$ finally we need to define the column vector $P$ (where we use equation 66 in the matrix form:

\[
\begin{align*}
\tilde{V}_p^k &= L_e (\sigma^k \mathcal{L}_\xi + \gamma_2 G^k_e \tilde{V}_p^k + K \sum_{l \neq k} \beta_{l,k} G_{p,l}^l \tilde{V}_p^l)) - L_{i,k} G^k_i \tilde{V}_i^k \\
\tilde{V}_i^k &= L_e \gamma_3 G_p^k \tilde{V}_p^k - L_{i,k} \gamma_5 G_i^k \tilde{V}_i^k \\
\tilde{V}_e^k &= L_e \gamma_1 G_p^k \tilde{V}_p^k \\
k &= 1, \ldots, N
\end{align*}
\]  

(66)

we now want to write the equations above in matrix form, for $k = 1, \ldots, N$. To do this, define $M_1$ as the block diagonal matrix:

\[
M_1(s) \in \mathbb{C}^{3N \times 3N}, \quad M_1 := M_1^1 \oplus \ldots \oplus M_1^N
\]

\[
M_1^k := \begin{pmatrix}
0 & -\gamma_1 G^k_p L_e^k & \gamma_2 G^k_e L_e^k \\
\gamma_3 G^k_p L_e^k & -\gamma_3 G^k_p L_e^k & 0 \\
\gamma_5 G^k_p L_e^k & 0 & 0
\end{pmatrix}
\]  

(67)

additionally define the block matrix $M_2$:

\[
M_2 := \begin{pmatrix}
M_2^{1,1} & M_2^{1,2} & \cdots & M_2^{1,N} \\
M_2^{2,1} & M_2^{2,2} & \cdots & M_2^{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
M_2^{N,1} & M_2^{N,2} & \cdots & M_2^{N,N}
\end{pmatrix} \in \mathbb{C}^{3N \times 3N},
\]  

(68)

finally we need to define the column vector $P$:

\[
P(s) := (\sigma^1 L_e^1 \mathcal{L}(\xi^1), 0, 0, \sigma^2 L_e^2 \mathcal{L}(\xi^2), \ldots, \sigma^N L_e^N \mathcal{L}(\xi^N), 0, 0)^T \in \mathbb{C}^{3N \times 1}
\]  

(69)

(where we use $\mathcal{L}$ to denote the “Laplace transform” of white noise). Now we can write equation 66 in the matrix form:

\[
\tilde{V} = \tilde{V}(s, K) := (\tilde{V}_p^1, \tilde{V}_i^1, \tilde{V}_e^1, \ldots, \tilde{V}_p^N, \tilde{V}_i^N, \tilde{V}_e^N)^T \\
\tilde{V} = (M_1 + M_2)\tilde{V} + P
\]  

(70)

We observe that matrix $M_1$ corresponds to local interactions between neural masses within each region $k = 1, \ldots, N$, $M_2$ describes the effect of intra-neuronal dynamics between pyramidal neurons of different regions and $P$ corresponds to the effect of the random input into the system. We can easily solve the above for $V$:

\[
V(s, K) = (I - (M_1 + M_2))^{-1} P
\]  

(71)

since we are only interested in the time series of the pyramidal neurons, we can select the relevant MEG spectra from above:

\[
V_{MEG}(s, K) := (\tilde{V}_p^1, \ldots, \tilde{V}_p^N)
\]  

(72)
now the naive cross-spectral matrix - see appendix-D - is:

\[ S_{MEG}(s, K) := \mathbb{E}[\overline{V_{MEG}}(\overline{V_{MEG}})^*] = (S_{MEG}^{u,v})_{u,v} \]  

(73)

and we can easily calculate the values above by remembering that \( \mathbb{E}[\mathcal{L}(\xi^u)\mathcal{L}(\xi^v)^*] = \delta_{u,v} \) (from definition in subsection-1.2.2) - a variant of this is performed in subsection-2.6. From this we can use appendix-D to work out the theoretical covariance matrix between regions \( u, v \). Note that for the purpose of computation, we often need to approximate the integral with a sum e.g

\[
COV_{u,v} = \int_{-\infty}^{\infty} S_{MEG}^{u,v}(2\pi if) df \\
\approx \sum_{r=1}^{n-1} \frac{\Delta f_r}{2} (S_{MEG}^{u,v}(2\pi if_1) + S_{MEG}^{u,v}(2\pi if_2)), \quad f_1 < f_2 < ... < f_n, \quad \Delta f_r := f_{r+1} - f_r
\]  

(74)

or with some equivalent approximation (we use the previous).

Here is an example plot for the theoretical power spectral density:

![Theoretical Power Spectral Density](image)

Figure 19: The theoretical power spectral density, where \( N = 90, B \cdot \cdot \cdot \) is from subsection-2.4, the parameters are from subsection-2.5 and \( K = 15 \) and we see that it has a peak in the alpha-frequency range.

We would like to remind the reader that the theoretical spectra and cross-variance (and thus cross-correlation) values that we calculate, are not the exact real model-derived values. To be able to perform such a derivation, we introduced errors such as:

1) Using a linear approximation (in equation-66). Its accuracy may depend on the displacement of \( V \) from \( \nabla \) i.e whether \( V^k_x - \overline{V^k_x} \approx e, i, p, \quad k = 1, ..., N \) is “large”.
2) Only being able to use a finite number of frequencies in calculating equation-74.
3) Rounding errors which are unavoidable when performing these calculations on the computer.

We think however that problem-1 could potentially be the most significant issue. We again remark that this approach is taken in papers such as [6], [10], [14], [28].

We previously mentioned that we use these linear approximations and theoretical spectra to save considerable computational time - as it takes much longer to (accurately)
calculate relevant properties from simulated data. We provide a specific example to illustrate this fact. Suppose we are using our model equation-63 with the standard parameters from subsection-2.5 with two regions and we want to calculate the simple cross-correlation between regions i.e:

\[
N = 2, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

There are two possible routes that guide us towards our desired value. We could use the results in this subsection to work out the theoretical cross-correlation between the regions. Otherwise, we could perform simulations of our model using appendix-C and then calculate an estimate of the cross-correlation. Note that the accuracy of these simulations (may) improve depending on how small \( \Delta t \) (time step) is, how large \( T \) (time during which we simulate the time series) is and if we take some sort of average over multiple simulations. With the standard parameters, the model had a supercritical Hopf bifurcation (appendix-B) at \( K \approx 54 \), in a analogous fashion to subsection-1.5. Therefore we only considered the region where the system is stable ( \( 0 \leq K < 54 \)). We plot a comparison:

![Plotting the cross correlation between the two regions of equations-63,75.](image)

Figure 20: Plotting the cross correlation between the two regions of equations-63,75. Varying \( K \), we plot this theoretical cross-correlation value and also the estimated cross-correlation for different values of the time step \( \Delta t \), duration \( T = 30 \) (fixed), via simulated results.

The time it took with our computational resources (see appendix-C) to plot each line on the figure above was 1 hour for \( \Delta t = 1e - 5 \) and 14 hours for \( \Delta t = 1e - 6 \). The other plots including the theoretical plot could be calculated within a fast time. A brief glance at the figure suggests that using larger time-step values gave much poorer results especially near the bifurcation. Furthermore we can perceive an advantage of using the theoretical derivation, as its curve is continuous with respect to varying \( K \) - which is what we would expect. To attempt to obtain a similar result with simulated results, we would need to perform numerous averages over multiple simulations - which would take even more time.
To put this into context, in the future we will be using our model with 90 regions \((N = 90)\) instead of the 2 regions in figure-20. It thus seem pragmatic that we might want to use the theoretically derived results.

### 2.4 Manipulating the Data

There were two types of data that were gathered that will be presented in this subsection. The first was real functional data, obtained via MEG scans of real people. The second type of data to be presented, was anatomical data that represented real pathways in human brains. Our general aim, is to combine this anatomical data with the model and then to make some comparison with the real data.

The data used was obtained in the same way as [14]. Here is a brief summary of this process:

1) 10 human subjects were convinced to perform MEG scans whilst in resting state.

2) For each subject, raw MEG data was obtained by measuring the magnetic field strength with a number of sensors near the head. Subsequently a beamformer was applied to this data to obtain estimated time series for 90 regions of the cortex (regions described in [14]).

3) For each region of each subject, we had a time series sampled at 250 Hz for 185 - 300 seconds (different subjects had scans for different durations). The signals were scaled to have mean 0.

4) For each subject, the data was filtered with a second-order Butterworth filter (appendix-E) leaving frequencies in \([7, 13]\) Hz. This was performed for three reasons: 4a) We are interested in alpha-rhythms 4b) To avoid aliasing, see appendix-E 4c) Many frequencies outside of this range was due to non-brain sources (see figure-21b).

Note that as the data is gathered from humans whilst they are in resting state, the time-series gleaned are stationary [3],[7].

Now we present an example of what time series of a typical subject of a typical region looks like we are presented looks like pre- and post-filtration. The following figure corresponds to pre-filtration:
Figure 21: An example of unfiltered (broadband) data, from subject 7, brain region 51.

a) A time series plotting real voltage against time.
b) The power spectral density estimate (using Welch’s method appendix-D) of the time series

We want to point out that some of the frequency spikes in figure-21b are due to contributions from outside the brain. For example the high values at very low frequencies (less than $5\,\text{Hz}$) are partly due to noise from the machine needed to perform the MEG measurement. The peak at $50\,\text{Hz}$ is the frequency of the UK mains electricity! The smaller peak at $20\,\text{Hz}$ are known as beta rhythms in the brain and originate from the motor cortices. This is one reason why we needed to filter the data.

After filtering the data presented in figure-21, we obtained

Figure 22: An example of filtered data of a brain region from subject 7, brain region 51.

a) A time series plotting real voltage against time.
b) The power spectral density estimate (using Welch’s method appendix-D) of the time series

In effect we were presented with 10 matrices, $90 \times \text{Sample}(i), i = 1, \ldots, 10$ - where $\text{Sample}(i)$ is a number corresponding to $250\times$ period of scan for subject $i$. We were then easily able to obtain a covariance and correlation matrix of the time series with respect to the 90 regions for each subject. Averaging these over the subjects, produced
a good real-life approximation of the covariance and correlation matrix of humans in resting state, with respect to our 90 regions. Here are heat maps of both:

![Heat Map](image1)

![Heat Map](image2)

Figure 23: a) The real average correlations between regions, b) The real average covariances between regions

We can immediately see the benefits of working with either matrix. Working with the covariance matrix (figure-23a) provides us the opportunity to work with the real amplitudes (variances) of specific regions. However normalising this to the correlation matrix (figure-23b) allows us to put more emphasis on the cross-nodal influences between regions, which have very little amplitude/variance.

Another method of improving a potential model via experimental means, is by measuring the physical strength of the neural pathways between the different regions that we considered before. This is useful as neuronal signals will be more readily propagated through bigger neural pathways. Known generally as the structural connectivity matrix [25], this corresponds to the values $B_{\cdot \cdot}$ in our model equation-63 - a multiplier for signals propagating from one local region to another. These values were obtained by a technique known as diffusion tensor imaging and via the description in [14]. This process provided us with a $90 \times 90$ symmetric matrix $B$, with scaled values in $[0, 1]$. The matrix is symmetric as the technique only measures the strength of the physical connection and we scale the values as there is no useful physical unit to connect them too. Recall that in the model, these values will be multiplied by the global coupling parameter $K$, so a relationship between the $B_{\cdot \cdot}$ is sufficient to be useful. We now present some facts about the elements of $B$ in the following figure:
As we can see from figure-24 the vast majority of connections between regions are incredibly small or non-existent - $B$ is a sparse matrix. A quick visual comparison of figures-23,24 should present the reader some hints that there may be some sort of connection between the two matrices.

2.5 Fitting the Model

The goal of this subsection is to combine the model with our anatomical matrix, then compare it to the real MEG data in some fashion. We then hope to find a dynamical working point to fit our model.

Before we attempt to fit the model, we first define our standard parameters that we fix for our model. These can be found out by experimental observation - and are close to the values described in [14]. For this section we take:

- $\rho_1 = 2, \rho_2 = 1$,
- $h_e = 4, h_i = 32$,
- $\kappa_e = 162.5, \kappa_i = 40.625$,
- $\gamma_1 = \gamma_2 = 128$,
- $\gamma_3 = \gamma_4 = 64$,
- $\gamma_5 = 0$,
- $p = 270$,
- $\sigma = 1$.

These all have physical units that can be discovered in [14]. It is of course a gross simplification to try and describe such macroscopic dynamics with such few parameters - but what is important, is that our model produces the correct qualitative form of brain dynamics, along with an option of modifying certain biological quantities as input parameter to observe what changed.

The main qualitative property that we want our model with a parameter set to fulfil, is an ability to emulate alpha-rhythms. Using our parameters from above, we can see that first property is satisfied from figure-19. It is thus reasonable to take these values as our parameter set for the rest of this section.
Additionally and interestingly, the model undergoes a supercritical Hopf bifurcation (appendix-B) when $K \approx 18.6$:

![Figure 25: For our model equation-63 with standard parameters, a plot of the greatest real part of its Jacobian eigenvalues against $K$. We have a supercritical Hopf bifurcation - appendix-B - when this value at $K \approx 18.6$. The system is unstable and has limit cycles when $K > 18.6$.](image)

In fact if you were to take a single region and plot a time series pre- and post-bifurcation, it would look very similar to figure-14.

Now we proceed with our aim to find a dynamical working point. We note that there are many possible ways that we might consider a metric allowing us to compare the model against theoretical results. We start by describing some potential options:

The first approach we take is to compare real/empirical and theoretical correlation matrices with each other. These are a good reservoir for information concerning the interactions between voltage fluctuations, as well as being an object frequently studied in computational neuroscience.

We obtain the $90 \times 90$ correlation matrix by averaging the sample correlation matrices for the 10 subjects - see subsection-2.4. We then vectorised the strict upper triangle part of this matrix. Denote these two quantities respectively by

$$
\Gamma^R \in \mathbb{R}^{90 \times 90}, \quad Y^R := (\Gamma^R_{1,2}, \ldots, \Gamma^R_{1,90}, \Gamma^R_{2,3}, \ldots, \Gamma^R_{2,90}, \ldots)
$$

Additionally we calculated the theoretical correlation matrix (subsection-2.3) and we vectorise this in the same way as we did to $\Gamma^R$. Noting that this quantity depends on $K$ - a free parameter which we can change as an variable, we denote these two properties respectively as:

$$
\Gamma^T(K) \in \mathbb{R}^{90 \times 90}, \quad Y^T(K) := (\Gamma^T(K)_{1,2}, \ldots, \Gamma^T(K)_{1,90}, \Gamma^T(K)_{2,3}, \ldots, \Gamma^T(K)_{2,90}, \ldots)
$$

So we want to compare both $\Gamma^R, \Gamma^T(K)$ which are matrices and $Y^R, Y^T(K)$ which are vectors. Obvious candidates for obtaining comparisons include the classic vector and matrix norms, which we will use.

We also consider the following three metrics, two of which were used in [14]. The
first is the coefficient of variation of the amplitudes, which is simply defined as:

$$||\Gamma_{COV}^R - \Gamma_{COV}^T(K)||_{COV} := |\Gamma_{COV}^R - \Gamma_{COV}^T(K)|$$

$$\Gamma_{COV} := \frac{std(v_1, \ldots, v_{90})}{mean(v_1, \ldots, v_{90})}, \quad v_k = \text{variance of region } k$$

again we can easily obtain the theoretical and real values as before. This measures the spatial dispersion of the strength of local alpha rhythms. Furthermore we define two correlation coefficient norms as:

$$||\Gamma^R - \Gamma^T(K)||_\rho := 1 - |\rho_{\Gamma^R, \Gamma^T(K)}|$$

$$||Y^R - Y^T(K)||_\rho := 1 - |\rho_{Y^R, Y^T(K)}|$$

the first is related to correlation coefficient between the real and theoretical upper-triangular covariance matrices, whilst the second is related to the correlation coefficient between the real and theoretical vectorised correlation matrices. Note that the definitions in the two previous equations above, do not satisfy the traditional mathematical definition of a norm - we just use this term to indicate we want to consider a “distance” or “error” between two different quantities.

Having chosen a specific metric, we then calculate the value of this metric for different values of $K$ in the stable regime (so $0 < K < 18.6$ from figure-25 ). We then fit the model with respect to $K$, by picking the $K$ such that the chosen distance between $\Gamma^R, \Gamma^T(K)$ or $Y^R, Y^T(K)$ is “minimal” e.g where $||\Gamma^R - \Gamma^T(K)||_{\text{Metric}}$ is minimal.

To summarise, we consider the following norms for our purpose:

$$\text{To compare } \Gamma^R, \Gamma^T(K) : \quad L^1, L^\infty, L^2, ||.||_{COV}, \rho$$

where $L^\cdot$ corresponds to the standard matrix norms (note as $\Gamma$ is symmetric then $||\Gamma||_1 = ||\Gamma||_\infty$). To compare the vectors we consider:

$$\text{To compare } Y^R, Y^T(K) : \quad L^1, L^2, L^\infty, \rho$$

where the $L^\cdot$ norms are the standard vector norms. Armed with these considerable resources of comparing theoretical results against real results, we plot many diagrams and see where we might fit the model:
Figure 26: Comparing real data with theoretically-derived quantities, respect to different metrics. In each plot we examine the normed difference against $K$, where $K$ is in the stable regime $0 < K < 18.6$. The norms we consider in the plots are:

- $a) \| Y^R - Y^T(K) \|_1$
- $b) \| Y^R - Y^T(K) \|_2$
- $c) \| Y^R - Y^T(K) \|_\infty$
- $d) \| \Gamma^R - \Gamma^T(K) \|_1, \| \Gamma^R - \Gamma^T(K) \|_\infty$
- $e) \| \Gamma^R - \Gamma^T(K) \|_2$
- $f) \| \Gamma^R - \Gamma^T(K) \|_\rho$
- $g) \| Y^R - Y^T(K) \|_\infty$
- $h) \| \Gamma^R - \Gamma^T(K) \|_{Coeff}$

Using the norms that we considered, we appear to be presented with two different options for fixing $K$ to fit our models:

Fit-1) Fix $K$ so that the model is somewhat near the bifurcation, as optimal $K \approx [17.0, 18.1]$ for figure-26a,b,c,d,e,f,h. Considering the model to be near the bifurcation is taken in conclusion in papers such as [3], [6], [7], [8], [13].

Fit-2) Fit $K = 12.6$ from figure-26g. This would be fixing $K$ around $\frac{2}{3}$ down the stable regime.

It seems rather convenient that most norms allow us to fit the model to be near the bifurcation. This makes the author suspicious and so we shall explore the evidence for this in more detail. As mentioned before, calculating the theoretical properties of a model via a linear approximation is a common technique e.g [6],[14], [28]. How effective and accurate this is for different values of $K$ is unknown. An answer to this question is clearly important - since we want to fit the model to a local minimum respect to a metric, as in figure-26. More specifically, whether the monotonicity of $\| f(Data) - f(Theoretical(K)) \|_y$ (y-axis of the plots in figure-26) corresponds to an equivalent result with real model-derived properties, is of paramount importance if this technique is to be successful.

The only possible method we have of checking this, is by simulating some time series, replacing the values of the theoretical quantities derived with the corresponding derivation from simulations and comparing them against the real data. In this scenario, we are of course limited by our maximum computing power (appendix-C) which we now fully exploit. Taking two different norms that gave us different fitting results, we consider the two metrics $\| Y^S(K) - Y^R \|_1, \| Y^S(K) - Y^R \|_\rho$.

Firstly we consider the result of figure-26a, which was a representative case where we would fix the correlation matrix near the bifurcation. Plotting to see the effect of replacing theoretical values with simulations:

37
Figure 27: Inspecting the effect of replacing the theoretical results with simulated results, when calculating the local minimum with respect to figure-26a. To calculate the simulated results, we used a time step $\Delta t = \frac{1}{10000}$ s for duration $T = 30$ s.

There are a few things that we can observe:

1) The simulated results would still suggest that we would want to fix the model “near” the bifurcation, since the function $K \rightarrow ||Y^R - Y^S(K)||_{1}$ decreases.

2) For lower values of $K$, the theoretical model and simulations provide very similar results when compared with the real data, with respect to this metric.

3) There is a strong possibility that the linear approximation is more erroneous near the bifurcation. One reason could be that there is a greater possibility of damped oscillations with a greater deviation from the fixed points (see appendix-B), in this region compared to further away from the bifurcation. The initial deviation to start this would be noise, which we assumed to be small in the linearisation. See figure-14b for an example of damped oscillations. Indeed the very large increase of $||Y^R - Y^T(K)||_{1}$ when $K$ increases through $18.0 < K < 18.6$ certainly seems incorrect. This makes the minimum of $K \rightarrow ||Y^R - Y^T(K)||_{1}$ look highly suspicious, which is a shame, since this was located in the interior of the stable regime for $K$.

To observe whether this issue was a problem specifically generated by this norm (within the realm of norms that implied fit-1), we performed a (restricted) calculation using the $L^2$ matrix norm - i.e we worked out the value of $||\Gamma^R - \Gamma^S(K)||_{2}$ for different $K$ and we compared this result with that of figure-26e. This limited comparison gave the same qualitative result as above (not shown).

We now hope to have more success with the same process repeated, with the correlation coefficient cross-correlation metric from figure-26g. This was the only metric in the fit-2 category.
Figure 28: Inspecting the effect of replacing the theoretical results with simulated results, when calculating the local minimum with respect to figure-26g. To calculate the simulated results, we used time step $\Delta t = \frac{1}{10000}$ s for duration $T = 30$ s.

We are surely disappointed with this result, the only thing the two plots have in common are that they are decreasing as $K$ increases through $0 < K \leq 12.6$. We attempt to find out why there is such a large difference between the plots, by looking at the scatter plots for low $K$:

Figure 29: Two scatter plots comparing $Y^R$ with a) $Y^T(4)$ b) $Y^S(4)$, to describe the x-coordinate point $K = 4$ on figure-28. We also include the line of best fit.

The explanation of the disparity between the plots for low $K$ on figure-28, is probably due to the fact that the terms of $Y^S(4)$ seem to be dominated by noise. Thus we think it likely that for this range, the theoretical cross-correlations $Y^T(K)$ are more accurate.

For larger $K$ near the bifurcation we have an even larger problem. For the same reasons as before, we think it is likely that the linear approximation causes a significant error in the calculation of $Y^T(K)$. This metric may be even more sensitive to this error, which may suggest why the minimum is further from the bifurcation. But
in this dreadful case, we are unsure that the values of $Y^S(K)$ are not too influenced by noise (at any stage) to vouch for the correct monotonicity of $\|Y^R - Y^T(K)\|_\rho$.

In this mysterious scenario, we have totally inconclusive results for suggesting whether we should use $K = 12.6$ (fit-2) as our dynamical working point for fitting the model. We simply don’t know whether figure-26g should be believed or not.

The prominent moral that we should draw from this subsection, is that one must take care when performing linear approximations, especially near bifurcations. This message is particularly important as this method has been previously used e.g [6], [10], [14], [28], and is likely to be used again. This could lead to incorrect claims that the brain is “critical”, or “close to instability” - the concept that humans are close to these limit cycles and having pathological brain activity. As discussed in subsection-2.3, there are practical advantages to being able to compute approximate theoretical values of model properties. A more accurate approximation (e.g higher order terms of the Taylor series) should be considered, if it does not result in the necessary derivations being too complicated.

Furthermore we have learnt that we must take care when analysing simulated results, since numerical simulations do not reveal the ground truth due to its finite nature and the presence of noise. The consequence of this, is that we cannot always rely on this method to check on whether linearisation introduces significant errors in our calculations, or whether other approximate, theoretically-derived model properties are correct.

2.6 Estimating Sigma

In [14] the authors were criticised for considering the variance of the white noise $\sigma^k_k = 1, ..., 90$ variance to be either 0, 1 for each region. To attempt to rectify this travesty, we consider in this subsection if there is any new and intelligent way of estimating $\sigma^k_k = 1, ..., N$ - where these values may be different for different $k$. Resources available to us are the real data and our theoretical derivations from subsections-2.3,2.4.

An immediate problem to outline is that a priori we do not know which value of $K$ to fix before we have found our $\sigma^k_k$. Previously in subsection-2.5, we considered $K$ as a free parameter - now we have both $K$ and the vector $\sigma$. In any case, we start from equation-71 and try to derive a useful relationship:

$$\hat{V}(s, K) = (I_{3N} - \hat{M}(s, K))^{-1}\hat{P}(s), \quad \hat{M}(s, K) \in \mathbb{C}^{3N \times 3N}, \hat{P}(s) \in \mathbb{C}^{3N \times 1}$$

defining the matrix

$$i, j = 1, ..., N, \quad C_{i,j} := (I_{3N} - \hat{M}(s))^{-1})_{1+3(i-1),1+3(j-1)}$$

we can rewrite

$$\hat{V}(s) = C(s, K)L(s)W, \quad L(s) := \text{diag}(L_E^1(s), ..., L_E^N(s)), W := (\mathcal{L}(\xi_1)\sigma^1, ..., \mathcal{L}(\xi_N)\sigma^N)^T$$

so we let

$$\hat{V}(s) = R(s, K)W, \quad R = C(s, K)L(s) = \begin{pmatrix} \leftarrow r_1 \rightarrow \\ \leftarrow r_2 \rightarrow \\ \vdots \\ \leftarrow r_N \rightarrow \end{pmatrix}, \quad R \in \mathbb{C}^{N \times N}, r_i \in \mathbb{C}^{1 \times N}$$
now perform the calculation:

\[ \mathbb{E}[\hat{V}(s)\hat{V}(s)^*] = R(s,K)WW^*R(s,K)^* = R(s,K)\mathbb{E}[WW^*]R(s,K)^* \]  

(86)

where the second equality comes from the fact that \( R(s,K) \) is deterministic. Now we note:

\[ WW^* = (L(\xi_i)L(\xi_j)\sigma^i\sigma^j)_{i,j=1,...,N} \]

(87)

and using our definitions of white noise from subsection-1.2.2, we have

\[ \mathbb{E}[\hat{V}(s)\hat{V}(s)^*] = R(s,K)\text{diag}(\sigma^2)^1,..., (\sigma^2)^N)R(s,K)^* = (r_u r_v^*)_{u,v=1,...,N} \]

i.e

\[ \mathbb{E}[\hat{V}(s)\hat{V}(s)^*] = (r_u r_v^*)_{u,v=1,...,N}\text{diag}((\sigma^2)^1,..., (\sigma^2)^N) \]

(88)

we are only interested in the diagonal of the above which is the vector

\[ S_{MEG}(s) = \text{diag}(|r_1|^2,..., |r_n|^2)\overline{\sigma}^2, \quad \overline{\sigma}^2 := \left( \begin{array}{c} (\sigma^2)^1 \\ (\sigma^2)^2 \\ \vdots \\ (\sigma^2)^N \end{array} \right) \]

(90)

manipulating this, we conclude with

\[ \overline{\sigma}^2 = R_{IDMat}(K,s)S_{MEG}(s), \quad R_{IDMat} := \text{diag}(1/|r_1|^2,..., 1/|r_n|^2) \]

(91)

where we have again emphasised the fact that \( K \) is not known or fixed.

So we have been able to produce a formula that expresses \( \sigma \) in terms of \( K \) and other fixed parameters. We can obtain the value of \( S_{MEG}(s) \) from our data in subsection-2.4. We can then calculate \( R_{IDMat}(K,s) \) theoretically. For every fixed \( K \), the ideal situation would be that the right-hand-side of this equation-91 would be the same for every \( s = 2\pi i \xi \). As this is not the case, a method for finding a \( K \) would be to calculate the (theoretical) values of \( r_1 = r_1(s = 2\pi i \xi, K),..., r_n = r_n(s = 2\pi i \xi, K) \) on the \( (s = 2\pi i \xi, K) \) plane for different values of \( K \). We then fix \( K \) such that the estimate in equation-91 is “most fixed” in some way for some varying values of \( s = 2\pi i \xi \).

To first choose which values of \( s = 2\pi i \xi \) we might consider, we note that we had previously filtered the real data, leaving frequencies in the interval \([7,13]\). Therefore we can extract a certain number of frequencies from this interval. We arbitrarily took 6 values spanning the whole interval: \( \xi = [7.8128, 8.7894, 9.7660, 10.7426, 11.7192, 12.6958] \text{Hz} \). Note that taking more values, does not significantly change the results to be obtained. Therefore we use these values as \( \xi \) in our plan of estimation. With these 6 frequencies, we use equation-91 to calculate 6 estimates for each \( \overline{\sigma}^2 \) which we respectively denote

\[ (\overline{\sigma}^2(K))_j = \left( \begin{array}{c} (\sigma^2)_j^1 \\ (\sigma^2)_j^2 \\ \vdots \\ (\sigma^2)_j^N \end{array} \right), j = 1, ..., 6 \]

(92)

accentuating that the estimate varies for each \( K \) (which has not been fixed yet). Now we want to find out for which \( K \) the values of \( (\sigma^2(K))_j, j = 1, .., 6 \) are “closest” to each
other. A simple way of considering this closeness, would be to consider the standard
deviation of the estimates, or more precisely the metric:

\[ \hat{\sigma}^2(K)_{STD} := \sum_{k=1}^{90} \text{std}(\sigma^2_k, \ldots, \sigma^2_N) \tag{93} \]

which we say depends on \( K \). A good value of \( K \) would give a low value on equation-93. So we plot \( \hat{\sigma}^2(K)_{STD} \) against \( K \) and cross our fingers in the hope that we have a minimum, so that we can choose a \( K \). Plotting this graph:

![Figure 30: Plotting \( \hat{\sigma}^2(K)_{STD} \) against \( K \)](image)

Our plan has again been thwarted as there is no non-trivial “minimum” as we would have liked.

Hope is not lost however, as we can deduce that picking different values of \( K \) do
not significantly change our estimate of sigma. We naturally define our estimate of \( \sigma \)
as the non-weighted average over the different frequencies in equation-93, i.e by

\[
\left( \hat{\sigma}^2(K) \right)_M = \frac{1}{6} \sum_{j=1}^{90} \begin{pmatrix}
(\sigma^2)_j^1 \\
(\sigma^2)_j^2 \\
\vdots \\
(\sigma^2)_j^N
\end{pmatrix} = \begin{pmatrix}
(\hat{\sigma}^2(K))_{M,1} \\
(\hat{\sigma}^2(K))_{M,2} \\
\vdots \\
(\hat{\sigma}^2(K))_{M,90}
\end{pmatrix}, \quad N = 90 \tag{94}
\]

again emphasising that we have not yet fixed \( K \). Valid values of \( K \) are in the non-trivial interval \((0, 18.6)\) (see figure-25). We pick two different values of \( K \) in our allowed range and compare the two estimates of \( \sigma \) that would be produced from both value of \( K \). Indeed we find out that the estimates do not change that much when varying \( K \):

\[
\frac{1}{90} \sum_{k=1}^{90} |\left( \hat{\sigma}^2(18) \right)_{M,k} - \left( \hat{\sigma}^2(10) \right)_{M,k}| = 0.001, \quad \max_{k=1,\ldots,90} \left| \left( \hat{\sigma}^2(18) \right)_{M,k} - \left( \hat{\sigma}^2(10) \right)_{M,k} \right| = 0.0057 \tag{95}
\]
even better, the “difference/error” between the terms is much smaller than their average value:

\[
\frac{1}{90} \sum_{k=1}^{90} \left( \hat{\sigma}^2(10) \right)_{M,k} = 0.0198, \quad \frac{1}{90} \sum_{k=1}^{90} \left( \hat{\sigma}^2(18) \right)_{M,k} = 0.0188 \tag{96}
\]
Two possible options of estimating $\sigma$ are to arbitrarily fix $K$ to work out the fixed values of $\sigma^K$, or to find this quantity by performing some sort of average over specific values of $K$. We present the results from performing the first method:

![Graph showing the relationship between $\hat{\sigma}^2(K)$ and $K$ for different values of $K$. The x-axis represents regions $k = 1, \ldots, 90$. The graph also plots a scaled version of the real amplitudes of the regions from subsection 2.4.]

Figure 31: A plot of $(\hat{\sigma}^2(K))_{M,k}$ for different values of $K$, the x-axis corresponds to regions $k = 1, \ldots, 90$. We also plot a scaled version of the real amplitudes of the regions from subsection-2.4

From this figure we observe that increasing $K$ decreases the values of $(\hat{\sigma}^2(K))_M$. This probably happens because increasing connectivity between different brain regions (increasing $K$), results in a greater propagation of the effects of the present variability between different regions. Therefore it seems plausible that this would be compensated by then estimating that inputted variability from the random external sources be lower, since we are comparing the effects to fixed real data.

Additionally we note from figure-31 that the estimates of $(\hat{\sigma}^2(K))_M$ seem roughly proportional, but not exactly proportional, to the real value of the amplitude/variance of the signals of the brain regions. Whilst this is a slightly boring result, it is pleasing that our estimate makes some sort of sense.
Appendices

A Linear Time–Invariant Systems

Engineers and physicists often like to model systems as a black box model, which consists of a time-dependent input, with a “black box” - which transforms the input in some way to produce some output. This approach is extensively used in modelling, see e.g [45]. Here we present the approach from [29] and describe some properties associated with this way of thinking.

A black box model can readily be visualised as:

![Diagram](image)

Figure 32: A simple conceptual example of a “black box” system, with input $u$, the transformation $\mathcal{T}$, and response/output $y$

The transformation is some mapping between function spaces. An important thing to note, is that while (in most cases) both $u(s), y(t)$ in the above depend on time, we purposefully do not use the same time variable for both, as they may not be related (the transformation may act on the entire function such as the Fourier transform). In much sloppily written literature, this subtlety is overlooked. Often further properties are desired for these black box systems; a common characteristic is requiring it to be Linear and Time-Invariant, denoted LTI:

\[
\forall a, b \in \mathbb{R}, \text{inputs } x(t), y(t) \\
\mathcal{T}(ax(\cdot) + by(\cdot)) = a\mathcal{T}(x(\cdot)) + b\mathcal{T}(y(\cdot)), \\
y(t) = \mathcal{T}(u(\cdot)) \implies y(t - a) = \mathcal{T}(u(\cdot - a))
\]  

- which is the situation that is used in all models in this thesis, stemming from the synaptic response function in subsection-1.2.3. Physicists and engineers also like to consider the impulse response function:

\[
h(t) := \mathcal{T}(\delta(s))
\]

which is the output of the Dirac delta function - also known as an impulse. This notation may irritate mathematicians, since the above may not be defined. However as $\mathcal{T}$ is often described as applying a convolution of a Schwartz function to the input, we can interpret the impulse response in the standard way that we consider the delta function to be a distribution or measure i.e

\[
y(t) = \mathcal{T}(u(s))(t) = \int_0^t u(t - s)d(s)ds = <u, d>, \quad d(s) \in \mathcal{S}(0, \infty),
\]

\[
\mathcal{T}(\delta(s)) = d(0) = <\delta, d>
\]

where we have used notation from distribution theory [9]. Of course this can be effortlessly ignored if, as engineers/physicists say, “it all works out”.

Actually engineers and physicists like to consider every LTI system as applying a
convolution. The “proof” of this can be seen in discrete finite time. If the input of a system is a sum of weighted impulses

\[ u(t) = \sum_{k=1}^{M} c_k \delta_{t-t_k}, \quad t_1 < \ldots < t_M, c_i \in \mathbb{R} \tag{100} \]

then we can apply equation-97 and obtain

\[ y(t) = T\left(\sum_{k=1}^{M} c_k \delta_{t-t_k}\right) = \sum_{k=1}^{M} c_k T(\delta_{t-t_k}) = \sum_{k=1}^{M} c_k h(t-t_k) = c * h, \quad c(k) := c_k \tag{101} \]

where * is the standard convolution. We can now deduce that the function \( d \) in equation-99 is the impulse response function \( h \) from equation-98. So we obtain the following relation

\[ T(u(s))(\tau) := \int_{0}^{\infty} u(\tau - s)h(s)ds \tag{102} \]

There is another property that engineers and physicists want LTI systems to posses, the so called Bounded-Input-Bounded-Output condition (denoted BIBO), which is the following implication:

\[ u(t) \in L^\infty \implies y(t) \in L^\infty \tag{103} \]

and this is an equivalent condition to requiring

\[ h \in L^1 \tag{104} \]

Note how this is held for our functions equations-11,17,62.

One advantage of considering a system as a “black box” model, is the ability for us to consider the “change” that the map in the black block provides. This is frequently done in physics and engineering, but the particular field that suits the need of our project is signal analysis e.g [29].

If we have an input to a system \( u(t) \) and an output \( y(t) \) we define the transfer function from \( u \) to \( y \) as

\[ T_{u,y}(s) := \frac{\tilde{y}(s)}{\tilde{u}(s)} \tag{105} \]

where \( \tilde{\cdot} \) refers to the Laplace transform.

Suppose we can write the above as a fraction of two polynomials, where the denominator is of greater degree than the numerator i.e

\[ T_{u,y}(s) := \frac{N(s)}{D(s)}, \quad n = \text{deg}(N) < \text{deg}(D) = d \tag{106} \]

then we can write the above as a sum of partial fractions

\[ T_{u,y}(s) := \sum_{k=1}^{d} \sum_{m=1}^{m_k} \frac{A_{k,m}}{(s-s_k)^m}, \quad A_{k,m} \in \mathbb{R} \tag{107} \]

where \( s_k \) is a pole of \( T \) with multiplicity \( m_k \). Working out the inverse Laplace transform now produces

\[ T_{u,y}(t) := \mathbb{I}_{[0,\infty)}(t) \sum_{k=1}^{d} \sum_{m=1}^{m_k} A_{k,m} \frac{t^{m-1}}{(m-1)!} e^{s_k t}, \quad A_{k,m} \in \mathbb{R} \tag{108} \]
The logic behind this computation, is that the values of $s_k$ present an intuition concerning the stability of our black box system. We observe from equations 105, 108, that if we had for any $k$ that $Re(s_k) > 0$, an impulse input i.e a large but quick input, might destabilise the output disproportionately. This may for example push the system far away from a fixed point.

Within the paradigm of LTI systems, we can observe this more concisely. If we now consider the transform in the same way as equation-102, then we observe that the Laplace transform of the transfer function is the impulse function i.e $T_{u,y}(s) = h(s)$. Without difficulty we see that equation-108 is integrable if and only if $Re(s_k) < 0, \forall k$. By looking at equation-104, this is the same as saying that a system is BIBO stable if and only if $Re(s_k) < 0, \forall k$.

In the case of subsection-1.5, the transfer functions we study are particularly useful. Considering equation-50 decomposed, we linearise the function $S$ and obtain

$$\dot{x} = Jx + B(x)$$

where $x$ is the deviation of our original dynamical variables around their fixed points and importantly $J$ is the four-dimensional Jacobian of the system that we want to solve. Note how this expression is similar to the Taylor approximation of multivariate functions. Deploying the Laplace transform in equation-109 results in

$$0 = \frac{B(x(s))}{sI - J}$$

and the poles of the modulus of this function are also the eigenvalues of $J$ - these are also the solutions of the poles of equation-55. This approach was used in [28] for neural mass models.

B  Supercritical Hopf Bifurcations and Time Change for Local Martingales

In the introduction of this thesis, we mentioned that two areas of theoretical mathematics are useful for examining neural mass models - dynamical systems and stochastic analysis. To emphasise the supreme joy of working with different areas of mathematics in unison, we present two results from these respective areas, that are used in this thesis.

Considering neuronal dynamics with the study of dynamical systems is essential to trying to understand their behaviour [18]. Indeed bifurcations can explain the qualitative change in dynamics that is so crucial to different brain states. Here we present the normal form of a system that goes through a supercritical Hopf bifurcation and briefly analyse it, from the very readable [40].

The dynamical system we present here is much simpler than the deterministic variants of equations-49, 63. However this is a useful comparison to make, as topologically it is the same of our model equations near the bifurcating fixed points.

There are three main coordinate systems that we could state this normal form in, we present all three versions:

$$\dot{X} = (\mu - X^2 - Y^2)X - \omega Y,$$

$$\dot{Y} = (\mu - X^2 - Y^2)Y - \omega X$$

(111)
\[ \dot{Z} = (\mu + i\omega - |Z|^2)Z, \quad Z := X + iY \] (112)
\[ \dot{r} = \mu r - r^3, \quad \dot{\theta} = \omega + br^2 \] (113)

which are our equations in Cartesian, complex and polar coordinates respectively. The bifurcation parameter is \( \mu \). Examining the Jacobian of equation-111 at the origin, provides us eigenvalues and stability properties:

\[ \lambda_{\pm} = \mu \pm i\omega \]

\[ \mu < 0 \text{ stable fixed point at } (0,0) \]
\[ \mu = 0 \text{ Hopf bifurcation at } (0,0) \]
\[ \mu > 0 \text{ unstable fixed point at } (0,0) \] (114)

where a Hopf bifurcation is described in [18].

In our idealised situation, when we are post-bifurcation (\( \mu > 0 \)) we can deduce from equation-113 that

1) A limit cycle will grow from point (0,0) radius 0 and will increase proportionally to \( r = \sqrt{\mu} \).
2) The period of the trajectories on the limit cycle is \( \approx \frac{2\pi}{\omega} + O(\mu) \).

[40] then goes on to note two things that occur in most systems that exhibit supercritical Hopf bifurcations:

1) The limit cycle is often elliptical not circular, with increased distorted shape as \( \mu \) increases.
2) Often \( \omega \) is not strictly independent of \( \mu \), therefore the trajectories of \( \lambda_{\pm} \) would not be straight when plotted in the complex plane for different values of our bifurcation parameter.

From equation-114 we can also observe that for \( \mu < 0, \mu \approx 0 \) near the bifurcation, we are in a situation where we have damped oscillations. This involves the fixed point still being stable, but attracting trajectories much more slowly than for lower values of \( \mu \). Meanwhile the angular velocity will stay (approximately) fixed, as can be seen from equation-113.

In any case, the important lesson to take for our models, is that the location of the eigenvalues in a complex plain is crucial to determining which qualitative state the model is in. If we have an eigenvalue on the right-hand side of the complex plane, we are in a limit cycle regime, whilst if all the eigenvalues are in the left-hand side of the complex plane, we are still in a stable regime characterised by stable fixed points and a lack of limit cycles. This is related to the result in appendix-A.

Local martingales are central in the study of stochastic integration - they are indeed the process for which we defined the stochastic integral (as an integrator) in [39]. Brownian motion was also an object of much attention, being a central part of stochastic differential equations as well as having many interesting properties (see [30],[39]). It is thus somewhat surprising that these two fundamental concepts are related, in that every continuous local martingale is a time-changed Brownian motion. We present a much weaker version of the proof, which we use for the purposes of this essay.

**Theorem:** If \( M_t := \int_0^t e^{\gamma s} \, dB_s, \gamma \neq 0 \) then \( M_t = B_{<M>_t} = B_{\frac{1}{2\gamma}(e^{\gamma t} - 1)} \)
Proof: From [39] we know \( \langle M \rangle_t = \frac{1}{2\gamma}(e^{2\gamma t} - 1) \). We note for \( \gamma \neq 0 \) that \( f(t) := \langle M \rangle_t \) is strictly increasing. Now let \( N_t := M_{f^{-1}(t)} \) \( t \geq 0 \) and we want to show that this is B.M. We easily observe that \( N_0 = 0 \).

For \( 0 \leq t_1 < t_2 < t_3 < t_4 \) note as \( f \) is strictly increasing \( f^{-1}(t_1) < f^{-1}(t_2) < f^{-1}(t_3) < f^{-1}(t_4) \). So we have \( N_{t_1} - N_{t_2} = \int_{f^{-1}(t_2)}^{f^{-1}(t_1)} e^{\gamma s} dB_s, N_{t_2} - N_{t_3} = \int_{f^{-1}(t_3)}^{f^{-1}(t_2)} e^{\gamma s} dB_s \) and \( N \) has independent increments follows from the definition of the stochastic integral with the fact that B.M also has independent increments \( (e^{\gamma s} \text{ is non-random}) \). Again by the construction of the stochastic integral we have that \( N_{t_2} - N_{t_1} \) is normally distributed, since B.M is normally distributed and its integrand is non-random. As \( N_{t_2} - N_{t_1} \in \mathcal{M}^2 \) then \( \mathbb{E}[N_{t_2} - N_{t_1}] = 0 \). By Ito’s Isometry in [33] we have
\[ \text{Var}(N_{t_2} - N_{t_1}) = \int_{f^{-1}(t_2)}^{f^{-1}(t_1)} e^{2\gamma s} ds = t_2 - t_1. \]

Therefore we have obtained \( N_{t_2} - N_{t_1} \sim \mathcal{N}(0, t_2 - t_1) \).

Overall we have shown that \( N_t \) is standard Brownian motion. Therefore \( N_{f(t)} = B_{f(t)}. \) 

A full proof of the general case can be found in [21] - stating that if \( M_t \) is a continuous local martingale then \( M_t = B_{<M>_t}. \)

C Programming, Plotting Figures and Model Simulations

In this appendix, we describe the nuts and bolts of creating this thesis. Since there was much programming and computational work within, it seems natural to dedicate a bit of space to describe how this was actually done, as well as giving an account of the authors travails in this matter.

We had a standard desktop computer: Intel Core2 Quad CPU Q9650, with cores 3.00GHz \( \times 4 \) and 3.8 GB RAM, as well as access to the UPF cluster (to be discussed in detail later). The operating system used was the fantastic Ubuntu, an open source operating system based upon (GNU)/Linux [41]. The thesis was typed out in TexMaker which was based upon the famous TeX core [2].

With respect to programming languages used, Matlab [27] was used for the entirety of section-2 and figure-1. Python [34] was used for the rest of section-1. This gave the author an excellent opportunity to look at the differences between open-source and proprietary software, as well as noting some advantages and disadvantages of either approach.

The fact that Python was used at all, was opined by some to be a fine investment of time in exploring one of the prime products of the ever-growing, community-run, free-source movement. Others were bewildered that Matlab didn’t monopolise the author’s use, as was the case for everybody else. Yet working with Python often felt fantastic and every related computation/figure was performed on the author’s computer. It was already installed in Ubuntu and scripts were directly executed from the command terminal. The only installations needed were the Numpy module [32] to perform scientific programming and the Matplotlib [16] module to perform plots.

Benefits of using Python consisted of the amazement felt that an open community
project could feasibly replace an expensive product; programming in a simple to read, fine-looking code; excellent documentation; as well as an eager online-community that always had time to help hapless stragglers. Unfortunately some of the optimisation algorithms were both difficult to use and often gave odd/wrong answers without any indication of this fact (used to numerically find the fixed points of equations-28,54). This could be highly irritating and led to figures such as:

![Figure 33: A previous plot of figure-17a](image)

After much persuasion, the author surrendered to peer pressure and reluctantly gave in to using Matlab. It had an easy to use interface, but the author often found the documentation difficult to understand. Whilst it had a short-cut allowing easy use of the UPF cluster, this function suddenly stopped working one day. Since it was proprietary software, our technician couldn’t access the code to explore this problem. Instead he had to contact MathWorks directly and wait for their reply, which was that we did not have the (extra) necessary license to use this function. They wanted more money. This was very irritating, indeed it is still a mystery how this option stopped functioning.

The author’s ability to use the UPF cluster was essential to being able to complete this thesis. It was used in figures-20,27,28,29 as well in multiple other plots or examinations that were not presented. The cluster, well-managed by our technician Jordi Varela, consisted of 24 copies of the following: 2 CPUs of Xeon E5-2650 2.00GHz, 8 cores each CPU, 64Gb RAM Memory, 300Gb local SAS disk. With connection to computers: 2X10Gbps Ethernet BCM57800 and 2X1Gbps Ethernet. These resources were divided into 368 arbitrary slots, so the unfortunate author had to partake these with other users via a complicated and optimised queueing system. He eventually developed a technique of hogging the cluster when nobody else was present.

Furthermore the author was pleased to have to learn the basics of parallel computing and remote programming (programming for external machines), which are two growing and important fields.

Together these resources were crucial in being able to plot figures-2728. Even with our advanced computing capacity, it still took a day to plot each of them.

All the figures in the project were relatively easy to plot using their respective programs. Matlab had its inbuilt plotting software and Python had the excellent Matplotlib module. R made a cameo appearance in figure-5 [35]. The outliers figures to
we want to write the above only in terms of first order differentials, so we write

$$\forall i, x_i(t) =: x_i, \quad g_i(t, x_1, ..., x_m) =: g_i$$

along with $m$ second order S.D.Es:

$$a_{1,1}x_1'' + a_{1,2}x_1' = f_1(x_1, ..., x_m) + g_1 \star \xi_1$$

$$a_{2,1}x_2'' + a_{2,2}x_2' = f_2(x_1, ..., x_m) + g_2 \star \xi_2$$

$$\vdots$$

$$a_{m,1}x_m'' + a_{m,2}x_m' = f_m(x_1, ..., x_m) + g_m \star \xi_m$$

$$a_{i,j} \in \mathbb{R}, \quad \xi_i \perp \xi_j \ i \neq j$$

we want to write the above only in terms of first order differentials, so we write

$$a_{1,1}y_1' + a_{1,2}y_1 = f_1(x_1, ..., x_m) + g_1 \star \xi_1$$

$$a_{2,1}y_2' + a_{2,2}y_2 = f_2(x_1, ..., x_m) + g_2 \star \xi_2$$

$$\vdots$$

$$a_{m,1}y_m' + a_{m,2}y_m = f_m(x_1, ..., x_m) + g_m \star \xi_m$$

$$x_1' = y_1, ..., x_m' = y_m$$

$$a_{i,j} \in \mathbb{R}, \quad \xi_i \perp \xi_j \ i \neq j$$

this does have the disadvantage of doubling the number of differential variables in our system. In differential notation, we use equation-7 and see

$$a_{1,1}dy_1 = -a_{1,2}y_1dt + f_1(x_1, ..., x_m)dt + g_1 dB_1 dt$$

$$a_{2,1}dy_2 = -a_{2,2}y_2dt + f_2(x_1, ..., x_m)dt + g_2 dB_2 dt$$

$$\vdots$$

$$a_{m,1}dy_m = -a_{m,2}y_mdt + f_m(x_1, ..., x_m)dt + g_m dB_m dt$$

$$dx_1 = y_1 dt, ..., dx_m = y_m dt$$

$$a_{i,j} \in \mathbb{R}, \quad B_i \perp B_j \ i \neq j$$

Suppose we are now presented with the values of $(x_1(0), ..., x_m(0)), (x_1'(0), ..., x_m'(0))$ and we want to simulate the values of $(x_1(T), ..., x_m(T))$. We divide the region $[0, T]$ into $N$ equal segments i.e

$$0 = t_0 < t_1 < ... < t_N = T, \quad \Delta t := t_{n+1} - t_n$$

So to describe the general simulation step, suppose that we are at time $t_n$ and we want to advance to time $t_{n+1}$. The first step involves calculating

$$x_1(t_{n+1}) = x_1(t_n) + y_1(t_n) \Delta t, ..., x_n(t_{n+1}) = x_n(t_n) + y_1(t_n) \Delta t$$

Now we talk about the main method we used to simulate results for the stochastic differential equations-19,27,50. To be able to do this, we employ the Euler-Maruyama method, which is a nice extension to our old friend, the Euler method. For more information, or for variations of this method see [23].
D  Power Spectral Density

The general idea of Fourier series in mathematics, is to express a periodic continuous function or an discrete signal, as a linear combination of complex exponentials. As these complex exponentials can be considered as (sine and cosine) waves, engineers and physicists decompose such functions in sums of frequencies - which have physical properties. This is one sort of analysis in the “Frequency Domain” which can also involve taking the Laplace, Z, standard Fourier transform of functions and signals.

We firstly describe the theoretical definition of the power spectral density, which can be thought of as the “strength” or “predominance” of a frequency in a probabilistic signal. We then define how most applied mathematicians, physicists and engineers calculate this, both theoretically then how an estimate of this quantity can be gathered from real data.

Note that all processes in this subsection are assumed to be real, have mean 0 and are also assumed to be stationary (same as data subsection-2.4 and our models).

We first define the most abstract and theoretical case. This is when we need to describe a method of considering a stochastic process in the frequency domain (i.e as some manipulation of a Fourier transform). The following is called the power spectral density of a stochastic process $X_t$:

$$ S_{XX}(\xi) := \lim_{T \to \infty} \mathbb{E}\left[\frac{1}{2} \frac{1}{\sqrt{T}} \int_{-T}^{T} X_t e^{-2\pi i \xi t} dt \right] = \lim_{T \to \infty} \mathbb{E}\left[\frac{1}{2} \frac{1}{\sqrt{T}} \int_{-T}^{T} X(t) \overline{\hat{X}(-\xi)} dt \right] $$

which can be defined when $X$ is not integrable. Note if we want to only consider the positive part of the stochastic process i.e $t \geq 0$, then we can set this limits $[0, T]$ in the integrals and delete the $\frac{1}{2}$ multiple - this can be done throughout the rest of this appendix. The reason that we take this definition is that for a fixed path of a function vanishing outside $[-T, T]$, Parseval’s theorem will be satisfied i.e

$$ Y_t := X_t \mathbb{1}_{[-T, T]}(t) \frac{1}{2T} \int_{-\infty}^{\infty} |Y(t)|^2 dt = \frac{1}{2T} \int_{-\infty}^{\infty} |\hat{Y}(\xi)|^2 d\xi $$

51
and this would agree with an “averaged” Fourier transform. We are attempting to 
define the squared modulus of a Fourier transform, for probabilistic functions.

Furthermore we can extend our definition to when we have two signals $X, Y$. We 
can easily define the cross-spectral density between the two as:

$$S_{X,Y}(\xi) = \lim_{T \to \infty} \mathbb{E}\left[ \frac{1}{2} \frac{1}{\sqrt{T}} \int_{-T}^{T} X_t e^{-2\pi i \xi t} dt \frac{1}{\sqrt{T}} \int_{-T}^{T} Y_s e^{-2\pi i \xi s} ds \right]$$  \hspace{1cm} (124)$$

where $\bar{\cdot}$ corresponds to the complex conjugate. This is clearly related to equation-122.

The key observation to perceive, is that for processes $X, Y$ that are co-stationary i.e 
the autocovariance function only depends on the time difference between the functions:

$$\mathbb{E}[X_{t+\tau}Y_t] = r_{X,Y}(\tau) \quad \forall t, \tau$$  \hspace{1cm} (125)$$

we have that the cross-covariance function and the spectral density functions are 
Fourier pairs! We show one implication:

$$S_{X,Y}(\xi)$$

$$= \lim_{T \to \infty} \mathbb{E}\left[ \frac{1}{2} \frac{1}{\sqrt{T}} \int_{-T}^{T} X_t e^{-2\pi i \xi t} dt \frac{1}{\sqrt{T}} \int_{-T}^{T} Y_s e^{-2\pi i \xi s} ds \right]$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} r_{X,Y}(t-s)e^{-2\pi i (t-s)\xi} ds dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-2T}^{2T} (2T - |u|) r_{X,Y}(u) \, du$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \lim_{T \to \infty} (2 - \frac{|u|}{T}) \mathbb{I}_{[2T,2T]}(u) r_{X,Y}(u) e^{-2\pi i u \xi} \, du$$

$$= \lim_{T \to \infty} \int_{-\infty}^{\infty} \mathbb{I}_{[-2T,2T]}(u) r_{X,Y}(u) e^{-2\pi i u \xi} \, du$$

$$= r_{X,Y}(\cdot)(\xi)$$  \hspace{1cm} (126)$$

This is known as (a version of) the Wiener-Khinchin theorem. We used Fubini’s 
Theorem and the dominated convergence theorem, so a sufficient condition to be able 
to perform these steps is $r_{X,Y}(\cdot) \in L^1$. Fortunately we can then use results in Fourier 
analysis [9] to know that we can obtain $r_{X,Y}$ from $S_{X,Y}(\xi)$ via the inverse transform if 
$S_{X,Y}(\cdot) \in L^1$. This is handy as we want to compare how different components of our 
signals interact with each other. Defining the cross-spectral matrix for a multivariate 
signal (defined on positive time) as the matrix

$$(S_{X^a, X^b}(\xi))_{a,b=1,\ldots,N}, \quad X(t) = (X^1(t), \ldots, X^N(t))^T \in \mathbb{C}^{N \times [0, \infty)}, \quad X^a X^b \text{ co - stationary}$$  \hspace{1cm} (127)$$

we can thus work out the covariance and thus correlation structure of a multivariate 
signal from its spectral density function - which is performed in equation-74.
by physicists and engineers e.g. [29],[45]. Provided with an N-dimensional, potentially probablistic signal

$$V(t) \in \mathbb{C}^{N \times [0,\infty)} \quad (128)$$

then taking the unofficial/naive Laplace transform (also known as the “Laplace transform”) denoted by \(\tilde{\cdot}\) and assuming of course that everything works, the cross spectral matrix of this signal is

$$\mathbb{E}[\tilde{V}(2\pi i \xi)(\tilde{V}(2\pi i \xi))^*] \in \mathbb{C}^{N \times N} \quad (129)$$

After much convincing, this is the definition that the author took when calculating equations-23,73. He still feels like a traitor.

Often we consider a simpler case than the previous as we are not provided with fully continuous functions, but a sequence of random-variables sampled from a stochastic process. This would involve looking at the time series

$$..., X_{-2\tau}, X_\tau, X_0, X_\tau, X_{2\tau}, ... \quad \tau > 0 \quad (130)$$

which is a sequence of random vectors derived from a stochastic process. In this scenario, we can consider the discrete case in equation-122 - we comfortably replace the integrals there with sums. So we consider the integrals to have point masses at specific points, that relate it to the Lebesgue integration interpolation:

$$S_{XX}(\xi) := \lim_{K \to \infty} \mathbb{E}\left[\frac{1}{2K} \left| \sum_{k=\left[-\frac{K}{\tau}\right]}^{\left[-\frac{K}{\delta}\right]} X_{k\delta} e^{-2\pi i k\delta t} \right|^2\right] \quad (131)$$

Again the result of equation-126 holds (the sequence is still stationary) and the power spectral density and the autocorrelation functions are Fourier-pairs. In the discrete stationary case, from equation-131, if we have that the auto-covariance function is absolutely convergent i.e

$$\gamma_X(h) := \mathbb{E}[X(t)X(t+h)], \quad \sum_{k \in \mathbb{Z}} |\gamma_X(\delta k)| < \infty \quad (132)$$

then we can use Fubini/change of summation so that the spectral density function is the Fourier transform of the auto-correlation function:

$$S_{XX}(\xi) = \delta^2 \sum_{k \in \mathbb{Z}} \gamma_X(k\delta)e^{(-2\pi i)(k\delta)\xi} \quad (133)$$

In fact, even if equation-132 is not satisfied, we can still define a (unique finite) measure on \([0,1]\) such that the auto-correlation function will be the Fourier transform of the measure. The spectral function would be the Radom-Nikodym derivative of this measure with respect to the Lebesgue measure, if this existed. This is known as the Herglotz Theorem and can be found in [42]. In this scenario we effortlessly calculate the inverse Fourier transform

$$\gamma_X(h) = \sum_{m \in \mathbb{Z}} S_{XX}(\frac{m}{\delta}) e^{2\pi i h \frac{m}{\tau}}, \quad h \in \delta\mathbb{Z} \quad (134)$$

and thus we have another case of the Wiener-Khinchin theorem.
We have considered an infinite theoretical time series on $\tau \mathbb{Z}$. Reflecting real-world situations of having a finite time series, we consider this theoretical variant where we have been given $n$ time-ordered observations sampled from a signal. These can be theoretically considered as $n$ ordered random variables:

$$
x_0, \ldots, x_{\tau(n-1)}, X_0, \ldots, X_{\tau(N-1)}
$$

we can of course observe how we could relate the above to equation-130, by infinitely repeating the finite sequence

$$
..., X_{-\tau}(:= X_{\tau(N-1)}), ..., X_0, ..., X_{\tau(N-1)}, X_{\tau N}(:= X_0), ...
$$

this is of course analogous to how we compute Fourier series by extending a function defined on a bounded interval to $\mathbb{R}$, by considering it to be periodic. In this scenario we use the finite Fourier transform on on the previous to analyse the frequency spectrum:

$$
\hat{X}(\xi \tau) = \tau \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} X_{t\tau} e^{-2\pi i t \xi / N}, \quad \xi = 0, \ldots, N - 1
$$

and the inverse finite Fourier transform can easily be seen here

$$
X_{k\tau} = \frac{1}{\tau} \frac{1}{\sqrt{N}} \sum_{\xi=0}^{N-1} \hat{X}(\xi \tau) e^{2\pi i k \xi / N}, \quad k = 0, \ldots, N - 1
$$

this is the same as the Fourier series of the interpolation of equation-136. We also note that we are performing operations on the group $\mathbb{Z}_N$ (modulo) - observing $e^{\pm 2\pi i x / N}$, $x \in \mathbb{Z}$ a solution for the $N$-th root of unity. Additionally the only frequencies of the of sample/time series (equations-135,136) that we need to consider - indeed that are necessary to perform the inverse Fourier transform - are the values that $\hat{X}(\cdot)$ takes in equation-137, which are the frequencies $0, \tau, ... (N-1)\tau$. Because of this the frequency $\tau$ is known as the fundamental Fourier frequency, with Fourier harmonics $\tau \times h, h = 2, 3, \ldots N - 1$.

Now we attempt to combine our theoretical results to justify some sort of way of estimating the power spectral density, which is used when we are presented with real data in subsection-2.4. We present the method called Welch’s estimate, with birthplace [43].

Suppose that we want to estimate the power spectral density at frequency $\lambda$ of our process $X$ sampled by equation-135. Looking at equation-122, the first estimator considered must surely be:

$$
I_{N,X}(\lambda)\tau = \frac{1}{N} \left| \tau \sum_{t=0}^{N-1} X_{t\tau} e^{-2\pi it \lambda / N} \right|^2, \quad \lambda = 0, \ldots, (N - 1)
$$

where we only need to consider the power spectral density at the Fourier frequencies as discussed. If we were have more and more observations ($N \rightarrow \infty$) then we “should” have a “approximation” to equation-122. Unfortunately it turns out that we may need to modify our definition of equation-139 to have better statistical properties see [42]. This is done by weighting the coefficients the transform in some fashion. Welch’s estimate performs this and we now present it.

Suppose that we have our variables from equation-135. Since the sampling frequency is $\frac{1}{\tau}$, we want to find potential frequencies up to $\frac{1}{2\tau}$ see appendix-E. We pick
total segments of even lengths \( L \) units, which start \( D \) units apart, which will cover the entire series, so \((K-1)D+L = N\). We define

\[
X^k(j) := X_{\tau(j+(k-1)D)}, \quad j = 0, ..., L - 1
\]

and performing the finite Fourier transform of each of these segments results in:

\[
A_k(n) = \frac{\tau}{L} \sum_{j=0}^{L-1} X^k(j) W(j) e^{-2\pi i j n L}, \quad n = 0, 1, ..., \frac{L}{2}
\]

where \( W(j) \) are data windows. Two suggestions presented in [43] for these are:

\[
W_1(j) = 1 - \left(\frac{j - \frac{L-1}{2}}{\frac{L+1}{2}}\right)^2, \quad W_2(j) = 1 - \frac{j - \frac{L-1}{2}}{\frac{L+1}{2}}, \quad j = 0, ..., L - 1
\]

Now we define the modified periodograms as

\[
I_k(f_n) = \frac{L}{U} |A_k(n)|^2, \quad k = 1, ..., K, \quad f_n = \frac{n}{\tau L}, \quad n = 0, ..., \frac{L}{2}
\]

\[
U = \frac{1}{L} \sum_{j=1}^{L-1} (W(j))^2
\]

and we average over these to find the spectral estimates:

\[
\hat{P}(f_n) = \frac{1}{K} \sum_{k=1}^{K} I_k(f_n), \quad f_n = \frac{n}{\tau L}, \quad n = 0, ..., \frac{L}{2}
\]

The programme that we use to calculate this value for our project, see appendix-C, picks as default

\[
K \approx 8, \quad D \approx \frac{L}{2} W(j) = 0.54 - 0.46cos\left(\frac{2\pi j}{L-1}\right)
\]

of course we can change these values in the program, if we wish.

E Nyquist–Shannon Sampling Theorem and Filters

In the theoretical universe and (as an almost perfect approximation) in the real world, we consider signals and waves to be defined on the continuum, i.e as functions

\[
f : \mathbb{R} \rightarrow \mathbb{R}
\]

which can often be periodic. Unfortunately for computational and practical reasons, we often can only think of waves taking values at discrete time points i.e

\[
..., f[-1], f[0], f[1], ... \quad f[n] := f(n\tau), \quad n \in \mathbb{Z}, \tau > 0
\]

which are realisations of the theoretical situation we considered in equation-130. Ideally we would want to find the frequency spectrum of the theoretical wave, which would be the Fourier transform if \( f \) is not periodic, or the Fourier coefficients of its Fourier series if \( f \) is periodic.
It would thus be an interesting question to ask whether the frequency spectrum of the theoretical signal would be changed after sampling and indeed if any information in the frequency spectrum is “modified” or “lost” in some fashion. This of course happens in real-world situations. It would not be surprising if this was the case, as we are now approximating a process of a finite number of points with trigonometric functions, instead of performing the same approximation of the original continuous signal. Here we present a sufficient condition for there not to be such a loss:

Nyquist-Shannon Sampling Theorem: Suppose we have a real signal \( f \in L^1 \) with (unitary) Fourier transform \( \hat{f} \in L^1 \) where \( \hat{f}(\xi) = 0 \) when \( |\xi| > \xi_b \), additionally we define \( f[i], i \in \mathbb{Z} \) as in equation-147 which has sampling frequency \( \xi_s = \frac{1}{T} \). If \( \xi_s > 2\xi_b \) then we can uniquely determine \( f(t) \) from the sample \( f[n] \) with the formula:

\[
f(t) = \sum_{n \in \mathbb{Z}} \frac{\sin(\xi\pi(t - nT))}{\xi\pi(t - nT)}
\]

Proof [29]: As \( f, \hat{f} \in L^1 \) we can use the Fourier inversion theorem to express

\[
f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i \xi t} d\xi = \int_{-\xi_s}^{\xi_s} \hat{f}(\xi)e^{2\pi i \xi t} d\xi
\]

where we also used the fact that \( \hat{f}(\xi) = 0 \) when \( |\xi| > \frac{\xi_s}{2} \). We can now express \( \hat{f} \) as a Fourier series on \([-\frac{\xi_s}{2}, \frac{\xi_s}{2}]\) (i.e we consider \( \hat{f} \) to be a periodic function on intervals \([k\frac{\xi_s}{2}, (k+1)\frac{\xi_s}{2}], \ k \in \mathbb{Z}\)) to obtain

\[
\hat{f}(\xi) = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{2\pi i n \xi \delta \xi}, \ \xi \in (-\frac{\xi_s}{2}, \frac{\xi_s}{2})
\]

which is the same as writing

\[
\hat{f}(\xi) = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{2\pi i n \xi \delta \xi} \mathbb{I}_{[-\frac{\xi_s}{2}, \frac{\xi_s}{2}]}(\xi), \ \xi \in \mathbb{R}
\]

where we can express the Fourier series coefficients as

\[
\hat{f}_n = \frac{1}{\xi_s} \int_{-\xi_s/2}^{\xi_s/2} \hat{f}(\xi)e^{-2\pi i \xi nt} d\xi = \frac{1}{\xi_s} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-2\pi i \xi nt} d\xi = \frac{1}{\xi_s} f(-nT) = Tf[-n]
\]

where we used the Fourier inversion of \( f \). Plugging this equation into the previous produces

\[
\hat{f}(\xi) = T \sum_{n \in \mathbb{Z}} f[-n]e^{2\pi i n \xi \delta \xi} \mathbb{I}_{[-\frac{\xi_s}{2}, \frac{\xi_s}{2}]}(\xi), \ \xi \in \mathbb{R}
\]

now we can take the inverse Fourier transform of each term of the previous to obtain our desired result. ■

Actually we can run into problems if we do not obey the premise of the theorem - namely that \( \xi_s \leq 2\xi_b \), called the Nyquist criterion. This is because it is possible that we could encounter a situation called aliasing, where a sampled signal may be expressed as non-unique sums of frequencies. We present such an example: suppose we sample a signal with period \( T \) and we find a continuous function \( f \) that describes this
discrete signal fully, then the function \( g(t) := f(t) + a \cdot \sin(2\pi \frac{t^2}{T}) \) will also describe the signal fully, for any \( a \in \mathbb{R} \). Another well-known interesting phenomenon caused by this effect, appeared in old western films and consisted in the viewer thinking that wagon wheels were moving with the opposite rotation to reality. This occurred because the frame rate of these films was less than twice the rotational frequency rate of the wheels.

It is thus very important that signals are both low-pass (or band-passed) filtered and sampled with a very high frequency.

We have just seen that we may want to filter the data between two frequencies to avoid aliasing. Furthermore we may want to delete certain frequencies from signals for a variety of regions, performed in subsection-2.4. This is called a bandpass filter and in effect, we would want to remove all frequencies that do not appear in our desired interval. Let’s discuss this theoretically. Suppose we have a real signal \( f \) as in equation-146. We wish to transform this function in a way that its frequency spectrum \( \hat{f} \) will be multiplied by a step function. So we seek a \( T \) satisfying

\[
\hat{T(f)} = \hat{f}(\xi) \times \mathbb{I}(\xi)_{[a,b]}
\]

By the property of the Fourier transform (also for the Laplace transform) we know that this is the same as finding a function \( h \) where

\[
\hat{T(f)} =: f \ast h, \quad \hat{h}(\xi) = \mathbb{I}(\xi)_{[a,b]}
\]

and taking the Fourier inverse of the above results in

\[
h = \frac{\sin((b-a)t)}{t} e^{2\pi i \frac{a+b}{2}}
\]

Examining this transform \( T \) and comparing it to concepts discussed in appendix-A, we see that \( T \) is not a bounded-input-bounded-output system as \( h \notin L^1 \) (then use equation-104). Therefore this theoretical filter is never used in practise.

A method to try settle this problem, is by using polynomial approximations of the step function within the frequency domain. There are many such filters that do this see e.g [21]. We describe the Butterworth transfer function of order \( n \) here. Suppose we have \( b = -a =: \xi_c \). We define the function \( H \) whose Laplace transform has amplitude

\[
|\hat{H}(2\pi i \xi)| = |\hat{H}(\xi)| = \frac{1}{\sqrt{1 + (\frac{\xi}{\xi_c})^{2n}}} \approx \mathbb{I}(\xi)_{[-\xi_c,\xi_c]}
\]

rewriting this in a more general Laplace Transform form we obtain:

\[
|\hat{H}(s)|^2 = \frac{1}{1 + (\frac{2\pi i s}{\xi_c})^n}
\]

and note that the above is the same as equation-151 when \( s \) is imaginary. By considering the \( 2n \)-th square roots of \( -1 \), the \( 2n \) poles of the above are

\[
|\hat{H}(s)|^2 = \xi_c e^{i\pi \frac{2k+1}{2n}} =: s_k, \quad k = 0, ..., 2n - 1
\]

so the idea is to use \( h(s) = \mathcal{L}^{-1}(\hat{H}(s) \ast \hat{H}(s)) \) as (an approximation to) the function we are looking for in equation-149. Returning to the definition of a transfer function
in appendix-A - which we consider \( h \) as being the convolution for - we want its poles to all have negative real parts (see equation-108). Therefore we ignore the \( n \) solutions of equation-153 that have positive real parts and let

\[
h(s) := \mathcal{L}^{-1}(\prod_i \frac{1}{s - s_i}), \quad \text{Re}(s_i) < 0
\]  

(154)
References


[34] Python Software Foundation, *version 2.7.6*, Open Source (GNU), 2013.


