HEDGING EUROMTS BOND INDEX FUTURES WITH EURIBOR FUTURES

By

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To whom I love.

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Abstract

Hedging strategies are one of three important elements for trading companies to survive in the current electronic trading environment, as good strategies definitely strengthen the company competitiveness. Current innovative financial products provide investors many possibilities to construct profit strategies and reduce their trading risk.

The aim of this work is to explore the possibility of hedging with Euribor futures a new interest rate derivative called EuroMTS Government Bond Index Futures. To do this, we adopts the two-additive-factor Gaussian model (G2++) to derive explicit pricing formulas of bond futures and Euribor futures. The calibration of the two-additive-factor Gaussian model has been done by using liquid fixed income products-Euro swaps, which shows that the G2++ model can replicate the market data nicely. The hedging strategy of Euribor futures and EuroMTS Bond Index futures has been tested to promote the usefulness of the G2++ model.

Keywords: Hedging strategies; The G2++ model; Euribor Futures; EuroMTS Bond Index Futures

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Introduction

During the past two decades, the market for contingent claims has experienced rapid growth and many innovative product creations. The financial market is becoming more competitive than before. Especially after January 2002, electronic trading has been introduced by Euronext.liffe. The intensive competition and thin margins of derivative trading leave some trading companies out of business and certainly create an opportunity for others. The fact that all trading companies have realized is that excellent human resources, modern information technology and advanced trading strategies are three essential elements in the current trading environment. One second is slow in the electronic trading environment, since the profitable opportunities shown on the screen might have been taken by competitors. Information technology plays an important role in the current trading. Good trading strategies support a trading company to edge other trading companies out, generate profit and support a long-term growth of it. Current innovative financial products provide equal profit opportunities for companies to design potential hedging strategies. This report focuses on some new bond derivatives in the fixed income market and explores the possibilities of a hedging strategies by using some of them.

The fixed income market is a financial market where participants buy and sell debt securities usually in the form of bonds. The size of the international bond market is an estimated \$45 trillion of which the size of outstanding U.S. bond market debt is \$25.2 trillion. Participants of bond markets are similar to participants in most financial markets and are essentially either buyers of funds or sellers of funds and often both. Participants include: institutional investors, governments,traders and individuals. Because of specific individual issues, and the lack of liquidity in many smaller issues, the majority of outstanding bonds are held by institutions like pension funds, banks and mutual funds. For bond market participants who own a bond, collect the coupon and hold it to maturity, market volatility is irrelevant; principal and interest are received according to a pre-determined schedule. But participants who buy and sell bonds before maturity, are exposed to many risks, most importantly changes in interest rates. When interest rates increase, the value of existing bonds fall, since new issues pay a higher yield. This is the fundamental concept of bond market volatility: changes in bond prices are inverse to changes in interest rates. Fluctuating interest rates are caused by many reasons. Some part is due to a country's monetary policy, as bond market volatility is a response to expected monetary policy and economic changes.

Market volatility of bond prices is generated by the difference between economists' consensus views of economic indicators and actual released data. A tight consensus is generally reflected in bond prices and there is little price movement in the market after the release of "in-line" data. If the economic release differs from the consensus view, the market usually undergoes rapid price movements as participants interpret the data. This is exactly what happens on the daily market activity. This economic analysis illustrates the intraday volatility. But from the mathematical angle, mathematicians observe that the interest rate also follows the mean reversion process due to the cycle of economic growth: when the economy grows fast, the central bank of a country will increase the interest rate to prevent the overheat of the country's economy; and when the economy slows down, the interest rate would be reduced to stimulate the investment and borrowing. The interest rate would fluctuate around a certain level in the long term. How this intriguing dynamics of interest rates can be systematically captured in a mathematical way is a challenge for mathematicians. Financial institutions employ many quantitative researchers to work on this issue, simply because the more accurately the models describe the term structure of interest rates, the less risk the fixed income investment based on those models might be exposed to. The choice of models truly depends on the practical situation, since no model is so powerful that it can be used in all situations.

In the current literature, there are many interest rate models. Well-known one-factor short rate models are the Vasicek model, the Dothan model, the Cox-Ingersoll-Ross model (CIR), the Hull-White model (HW) and the Black-Karasinski model (BK). Besides these, there are two-factor models such as the two-additive-factor extended CIR model (CIR++) and the two-additive-factor Gaussian model (G2++). In this project, we aim to use the G2++ model to find the pricing formulas of bond futures and Euribor futures and then investigate whether it is possible to hedge bond futures with Euribor futures and how good these hedging strategies will be.

The rest of this thesis is structured as follows: the financial products we are interested in this project are described in Chapter 1; the motivation of the two-factor-additive Gaussian model in this project is explained in Chapter 2; and then we present the mathematical dynamics of the two-additive-factor Gaussian model. Based on it, we derive the pricing formulas of bond futures, Euribor futures and total return bond futures in Chapter 4. In Chapter 5, we present the different methods of calibration and how we calibrate the G2++ model to the real market data. Chapter 6 is about theoretical hedging between EuroMTS Bond index futures and Euribor futures. The hedging results are also presented.

Chapter 1

Futures

In this project, we are especially interested in futures contracts. Futures, as one of the most developed financial products, can be categorized into different types based on their underlying. In this chapter, we present an introduction of futures and familiarize readers with what we are using in this paper.

1.1 A Short History of Futures

In finance, a futures contract is a standardized contract, traded on a futures exchange, to buy or sell a certain underlying instrument at a certain date in the future, at a specified price. A futures contract gives the holder the obligation to buy or sell, which differs from an options contract, by which the holder has the right , not the obligation. The origins of futures trading can be traced back to Ancient Greek or Phoenician times. But the modern futures trading begins in Chicago, the United States in the early 1800s. In 1848, the Chicago Board of Trade(CBOT)– the world's first futures exchange–was established. Trading was originally in forward contracts. In 1865, standardized futures contracts were introduced. With the development of the economy, many different types of futures were developed to satisfy the need of investors. In 1972, currency futures were introduced by the International Monetary Market(IMM). Later in the 1970s, the financial futures contract allowed trading in the future value of interest rates. The current futures markets are so developed that anything, which is easily kept and rare, can be taken as an underlying to launch futures contracts. Currently, greenhouse gas futures are getting popular. Based on different underlyings, futures generally can be categorized into currency futures, interest rate futures, equity futures, commodities futures and they can be classified into subcategories. In this project, our interest lies in interest rate futures.

1.2 Interest Rate Futures

An Interest Rate Future is a futures contract with an interest-bearing instrument or rate as the underlying asset. Examples include Treasury-bill futures, Treasury-bond futures and Eurodollar futures. The latest launch of EuroMTS bond index futures are particularly interesting since these are the first interest rate futures that give investors similar properties to equity index futures. To hedge EuroMTS bond index futures with Euribor futures is the subject of this work.

1.2.1 Euribor Futures

Euro Interbank Offered Rate, abbreviated as Euribor, is a daily reference rate based on the averaged interest rates at which banks offer to lend unsecured funds to other banks in the euro wholesale money market. Euribor rates are used as a reference rate for euro-dominated interest derivatives like forwards, futures and swaps. They are the basis for some of the world's most liquid and active interest rate markets.

Euronext.liffe's Euribor futures contracts, as the world's second most heavily traded short term interest rate futures contracts, are based on three-month Euribor rates.

1.2.2 EuroMTS Bond Index and Futures

The EuroMTS Bond Index originated from CNO Etrix indices, which were developed by the Comité de Normalisation Obligataire (French Bond Association) in 1989. At the beginning, it only included French Sovereign debt and later evolved to include eurozone government bonds from 1st January 1999. In January 2003, EuroMTS Ltd acquired the intellectual property right of it. From then on, the EuroMTS index is marketed and developed by MTSNext, a joint venture of EuroMTS and Euronext.

Among them, we are especially interested in the EuroMTS Eurozone Government Index, which is a euro-denominated total return index series which measures the performance of the eurozone government bond market. The EuroMTS Government Index is composed of all the eurozone government bonds listed on the MTS platforms with more than EUR 2 billion outstanding volume and at least one year to maturity. The series comprises 6 maturity bands sub-indices and is split in several country index families.

The MTS France Government Bond Index, MTS German Government Bond Index and MTS Italian Government Bond Index are especially interesting, since on 13th November 2006, Euronext.liffe announced the launch of a range of EuroMTS Government Bond Index Futures with those 3 Government Bond Index as underlying. The maturities of the underlying bonds range from 7 to 10 years.

Chapter 2

Motivation And Model Choice

The two-additive-factor Gaussian model has several advantages over the classical one-factor short rate models. These advantages can capture what have been observed in the fixed income market better. Certainly, the corresponding risk would be lower if we make decisions based on it. In the following, we will elaborate on the reason why we choose the G2++ model for this project. The precise formulation of the two-additive-factor Gaussian model will be introduced in Chapter 3.

2.1 Limitations of One-factor Short Rate Models

The pioneering short-rate model, the Vasicek model, was introduced in 1977 by Vasicek(1977). His work stimulated the further development in short rate models. With the growth of the fixed income market and the development of the interest rate theory, researchers gradually realized that there are some flaws with short-rate models, which contradict market facts. The adoption of the G2++ model in this project is due to the weakness of one-factor short rate models.

Short rate models such as the Hull-White model, the Black-Karasinski model, and the Cox, Ingersoll and Ross model prove useful when products which depend on a single rate for the whole interest-rate curve, are priced. If the correlation of different rates plays a role in the pricing, those models are insufficient. The reason is that all one-factor interest rate models assume that the correlation between instant rates of all maturities is equal to 1, which is unrealistic.

In order to show this, we need to look into them in detail. Take the Vasicek model for instance. The formulation of this model on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{Q})$, is

$$dr_t = k(\theta - r_t)dt + \sigma dW_t.$$

where r_0, k, θ and σ are positive constants, and $\{W_t, t \leq 0\}$ is a Brownian motion,under the risk neutral measure of $\mathbb{Q}$.

From this interest rate dynamics, we would derive zero-bond prices

$$P(t,T) = E^{\mathbb{Q}}[e^{-\int_t^T r_u du}|\mathcal{F}_t] = A(t,T)\exp(-B(t,T)r_t)$$

where

$$\begin{aligned} A(t,T) &= \exp\left\{(\theta - \frac{\sigma^2}{2k^2})[B(t,T) - T + t] - \frac{\sigma^2}{4k}B(t,T)^2\right\}, \\ B(t,T) &= \frac{1}{k}[1 - e^{-k(T-t)}]. \end{aligned}$$

From this zero-coupon bond pricing formula, we can find the continuously compounded spot rates

$$R(t,T) = -\frac{\ln P(t,T)}{T-t} = -\frac{\ln A(t,T)}{T-t} + \frac{B(t,T)}{T-t}r_t = a(t,T) + b(t,T)r_t.$$

If we are considering a payoff which is based on the joint distribution of two rates at different times, for example at $T_1 = t+1$ and $T_2 = t+5$, then the correlation between such two rates is really important since the joint distribution is involved. With the Vasicek model, the correlation can be computed

$$\mathrm{Corr}\big(R(t,T_1),R(t,T_2)\big) = \mathrm{Corr}(a(t,T_1)+b(t,T_1)r_t, a(t,T_2)+b(t,T_2)r_t) = 1,$$

which means that the instant rates for all maturities are perfectly correlated. In this example, it means that a shock to the interest rate curve at time t is transmitted through all maturities and the whole curve moves in the same direction when its initial point is shocked. Apparently, this contradicts the observed fact that interest rates are known to exhibit non-perfect correlation.

In this project, we need to hedge bond futures with Euribor futures with different maturities. So it is necessary to find a model that takes the non-perfect correlation of the interest rate curve into account.

2.2 Sufficiency of Two Factors

From Econometric analysis, we know that a model with more explanatory variables fits more data than others with less explanatory variables. Along the same lines, it definitely improves our model if we include more factors into the model. It is logical to ask a question how many factors we should use for practical purposes. When we determine how many factors we use in our model, we should make a good balance between numerically-efficient implementation and capability of the model to represent realistic correlation pattern. Fortunately, some researchers have done this principal component analysis.

According to Jamshidian and Zhu(1997), two factors can explain 85% to 90% of variation in the yield curve based on factor analysis with respect to JPY, USD and DEM data. In their research, it is shown that one principal component explains from 68% to 76% of the total variation, whereas 93% to 94% can be explained by 3 principal components. One more optimistic analysis done by Rebonato(1998) on the UK market shows that one component explains 92% of the total variance and comparably tow components already explain 99.1% of the total variance. With respect to the tractability and implementability of the models, we will focus on the two-factor-additive Gaussian model (G2++) in this project.

2.3 Advantages of The G2++ Model

The G2++ model is an interest rate model with two correlated Gaussian factors and one deterministic function that is used to calibrate the current interest rate curve, which will be formulated in Section 3.

The Gaussian properties of this model improve its convenience. Generally, Gaussian models are of good use in practice, since under the assumption of the normal distribution on rates many explicit pricing formulas of interest rate derivatives such as European options on pure discount bonds, caps and floors can be derived. Another advantage of the G2++ model, which one-factor models lack, is that the introduction of the correlation coefficient in the model helps to capture the decorrelation property of the interest rate curve.

Chapter 3

Short Rate Dynamics

In this chapter we present the dynamics of the two-additive-factor Gaussian models for the instantaneous rate. The second part of this chapter is mainly concerned with the pricing formula of zero coupon bonds, since the prices of zero coupon bonds are the fundamental information in the pricing of other fixed income products like Euribor futures and EuroMTS Bond index futures.

3.1 The G2++ Dynamics

We assume that under a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{Q})$, where $\mathbb{Q}$ is the risk neutral measure, the instantaneous-short-rate process under $\mathbb{Q}$ is

$$r(t) = x(t) + y(t) + \varphi(t),\ r(0) = r_0 \tag{3.1.1}$$

where the processes $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ satisfy

$$dx(t) = -ax(t)dt + \sigma dW_1(t),\ x(0) = 0$$
$$dy(t) = -by(t)dt + \eta dW_2(t),\ y(0) = 0$$

where (W_1, W_2) is two-dimensional Brownian motion with instantaneous correlation ρ so

$$d < W_1, W_2 >= \rho dt,$$

where r_0, a, b, σ, η are positive constants, and $-1 < \rho < 1$. The sigma algebra $\{\mathcal{F}_t\}$ is generated by $\{(x(s), y(s)), 0 \le s \le t\}$.

From this dynamics, it is easy to derive the instantaneous rate as follows:

$$x(t) = x(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_1(u), \tag{3.1.2}$$

$$y(t) = y(s)e^{-b(t-s)} + \eta \int_s^t e^{-b(t-u)} dW_2(u). \tag{3.1.3}$$

for $t > s \ge 0$, so we have

$$r(t) = x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_1(u) + \eta \int_s^t e^{-b(t-u)} dW_2(u) + \varphi(t).$$

As the two Brownian motions W_1 and W_2 are defined to be correlated, it may be difficult to generate their paths in the practice. If we introduce two independent Brownian motions $\tilde{W}_1$ and $\tilde{W}_2$ and define:

$$dW_1(t) = d\tilde{W}_1(t), dW_2(t) = \rho d\tilde{W}_1(t) + \sqrt{1-\rho^2} d\tilde{W}_2(t)$$

then W_1 and W_2 are as required above. In this way, the generation of two correlated Brownian motions are transferred to two independent Brownian motions. Moreover, the instantaneous interest rate equals

$$\begin{aligned} r(t) =& x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \sigma \int_s^t e^{-a(t-u)} d\tilde{W}_1(u) + \eta\rho \int_s^t e^{-b(t-u)} d\tilde{W}_2(u) \\ & \eta\sqrt{1-\rho^2} \int_s^t e^{-b(t-u)} d\tilde{W}_2(u) + \varphi(t). \end{aligned}$$

3.2 The pricing of Zero Coupon Bonds

In the fixed income world, the price of a zero-coupon bond plays an important role in pricing coupon bonds, since all the coupon bonds can be treated as a collection of zero-coupon bonds. It is worthwhile to give a precise definition of zero-coupon bonds.

Definition 3.2.1. (Zero-coupon bond) A T-maturity zero-coupon bond (pure discount bond) is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t, T)$.

Under the assumption of stochastic interest rates, the price $P(t, T)$ at time t of a zero coupon maturing at T under the risk neutral measure $\mathbb{Q}$ is

$$P(t, T) = E^{\mathbb{Q}}\{e^{-\int_t^T r_s ds}|\mathcal{F}_t\}.$$

In order to derive the formula of a zero coupon bond, we need to find the instantaneous short rate dynamics. We need the following lemmas.

Lemma 3.2.1 (Brigo and Mercurio)**.** *For each $t < T$, the random variable*

$$I(t, T) = \int_t^T [x(u) + y(u)]du$$

conditional on the sigma-field $\mathcal{F}_t$ is normally distributed with mean $M(t, T)$ and variance $V(t, T)$, respectively given by

$$M(t, T) = f_a(T - t)x(t) + f_b(T - t)y(t),$$

and

$$\begin{aligned}V(t,T) =&\frac{\sigma^2}{a^2}[T-t-2f_a(T-t)+f_{2a}(T-t)]+\frac{\eta^2}{b^2}[T-t-2f_a(T-t)\\&+f_{2a}(T-t)]+2\rho\frac{\sigma\eta}{ab}[T-t-f_a(T-t)\\&-f_b(T-t)+f_{a+b}(T-t)],\end{aligned}$$

in which $f_x(t) = \frac{1-e^{-xt}}{x}$.

Proof. By Ito's formula we know that for continuous semimartingales X and F,

$$d[F(t)X(t)] = F(t)dX(t) + X(t)dF(t) + d < X, F > (t),$$

Hence, we have

$$d[t \cdot x(t)] = t \cdot dx(t) + x(t) \cdot dt,$$

since $< t, x > (t) = 0$. By taking the integral, we obtain within the time interval [t,T]

$$\int_t^T d[u \cdot x(u)] = \int_t^T udx(u) + \int_t^T x(u)du.$$

So

$$\begin{aligned}\int_t^T x(u)du &= Tx(T) - tx(t) - \int_t^T udx(u)\\&= Tx(T) - Tx(t) + Tx(t) - tx(t) - \int_t^T udx(u)\\&= \int_t^T Tdx(u) - \int_t^T udx(u) + (T-t)x(t)\\&= \int_t^T (T-u)dx(u) + (T-t)x(t).\end{aligned}$$

Now we calculate the first term in the last equation by substituting the expression for $dx(u)$.

$$\int_t^T (T-u)dx(u) = -a\int_t^T (T-u)x(u)du + \sigma\int_t^T (T-u)dW_1(u).$$

By Equation 3.1.2 and 3.1.3, we also have

$$\begin{aligned}\int_t^T (T-u)x(u)du =& x(t)\int_t^T (T-u)e^{-a(u-t)}du \\ &+ \sigma \int_t^T (T-u)\int_t^u e^{-a(u-s)}dW_1(s)du.\end{aligned}$$

Now we calculate the last two terms above separately by multiplying $-a$

$$-ax(t)\int_t^T (T-u)e^{-a(u-t)}du = -x(t)(T-t) + f_a(T-t)x(t).$$

And

$$\begin{aligned}&-a\sigma\int_t^T (T-u)\int_t^u e^{-a(u-s)}dW_1(s)du \\ =&-a\sigma\int_t^T (\int_t^u e^{as}\underset{\cdot}{W}_1(s))d_u(\int_t^u (T-v)e^{-av}dv) \\ =&-a\sigma[(\int_t^T e^{au}dW_1(u))(\int_t^T (T-v)e^{-av}dv) \\ &-\int_t^T (\int_t^u (T-v)e^{-av}dv)e^{au}dW_1(u)] \\ =&-a\sigma\int_t^T (\int_u^T (T-v)e^{-av}du)e^{au}dW_1(u) \\ =&-\sigma\int_t^T [(T-u)+\frac{e^{(T-u)}-1}{a}]dW_1(u).\end{aligned}$$

From above, we obtain

$$\int_t^T x(u)du = f_a(T-t)x(t) + \sigma\int_t^T f_a(T-u)dW_1(u),$$

similarly, we also have the following

$$\int_t^T y(u)du = f_b(T-t)y(t) + \eta\int_t^T f_b(T-u)dW_2(u).$$

Thus, $M(t,T)$ is verified. Now we calculate the variance conditional on $\mathcal{F}_t$

$$\begin{aligned}
\mathrm{Var}\{I(t,T)|\mathcal{F}_t\} =& \mathrm{Var}\{\frac{\sigma}{a}\int_t^T [1-e^{-a(T-u)}]dW_1(u) + \frac{\eta}{b}\int_t^T [1-e^{-b(T-u)}]dW_2(u)|\mathcal{F}_t\} \\
=& \frac{\sigma^2}{a^2}\int_t^T [1-e^{-a(T-u)}]^2 du + \frac{\eta^2}{b^2}\int_t^T [1-e^{-b(T-u)}]^2 du \\
& + 2\rho\frac{\sigma\eta}{ab}\int_t^T [1-e^{-a(T-u)}][1-e^{-b(T-u)}]du \\
=& \frac{\sigma^2}{a^2}[T-t+\frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}] \\
& + \frac{\eta^2}{b^2}[T-t+\frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}] + \\
& 2\rho\frac{\sigma\eta}{ab}[T-t+\frac{e^{-a(T-t)}-1}{a} + \frac{e^{-b(T-t)}-1}{b} - \frac{e^{-(a+b)(T-t)}-1}{a+b}] \\
=& \frac{\sigma^2}{a^2}[T-t-2f_a(T-t)+f_{2a}(T-t)] + \frac{\eta^2}{b^2}[T-t-2f_b(T-t) \\
& + f_{2b}(T-t)] + 2\rho\frac{\sigma\eta}{ab}[T-t-f_a(T-t)-f_b(T-t) \\
& + f_{a+b}(T-t)]
\end{aligned}$$

□

In order to simplify our notation, we will use the following function also for the rest of our discussion

$$f_x(t) = \frac{1-e^{-xt}}{x}.$$

From Equation 3.1.1, we see that the instantaneous-interest-rate is not only determined by $x(t)$ and $y(t)$, but also by a deterministic function φ. The next corollary will tell us how we could determine this function from the market data.

Before we present the corollary, some notation should be introduced. Suppose that we can observe the term structure of discount factors in the market. And the smooth function $T \to P^M(t,T)$ can be found which fits this curve. The instantaneous forward rate $f^M(0,T)$ at time 0 for a maturity T can then be derived from $P^M(0,T)$, since

$$f^M(0,T) = -\frac{d\ln P^M(0,T)}{dT}.$$

Now we are ready to present the following corollary.

Corollary 3.2.2 (Brigo and Mercurio)**.** *The model fits the currently-observed term structure of discount factors if and only if, for each T,*

$$\begin{aligned}\varphi(T) =& f^M(0,T) + \frac{\sigma^2}{2a^2}(1-e^{aT})^2 \\ & \frac{\eta^2}{2b^2}(1-e^{-bT})^2 + \rho\frac{\sigma\eta}{ab}(1-e^{-aT})(1-e^{-bt})\end{aligned}$$

i.e., if and only if

$$exp\Big\{-\int_t^T \varphi(u)du\Big\} = \frac{P^M(0,T)}{P^M(0,t)}exp\Big\{-\frac{1}{2}[V(0,T)-V(0,t)]\Big\},$$

so that the corresponding zero-coupon-bond prices at time t are given by

$$\begin{aligned}P(t,T) &= \frac{P^M(0,T)}{P^M(0,t)}exp\{A(t,T)\} \\ A(t,T) &= \frac{1}{2}[V(t,T)-V(0,T)+V(0,t)] - f_a(T-t)x(t) - f_b(T-t)y(t).\end{aligned}$$

The above corollary will be used in the pricing of bond futures. The following proposition is of good use in the whole project, since we always switch between lognormal distributions and normal distributions, which is an important trick in the derivation of pricing formulas.

Proposition 3.2.3. *Let X be a random variable that is lognormally distributed, and the mean and variance of $Y = ln(X)$ are denoted by M and V respectively . Then*

$$E\{X\} = e^{\frac{V}{2}+M}.$$

Proof.

$$\begin{aligned} E\{X\} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V}} e^{y} e^{-\frac{(y-M)^2}{2V}} dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V}} e^{y-\frac{(y-M)^2}{2V}} dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V}} e^{\frac{-[y-(V+M)]^2+((V+M)^2-M^2)}{2V}} dy \\ &= e^{\frac{((V+M)^2-M^2)}{2V}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V}} e^{\frac{-[y-(V+M)]^2}{2V}} dy \\ &= e^{\frac{V}{2}+M}. \end{aligned}$$

□

From the proposition above, the pricing formula of one zero-coupon bond is concluded easily.

Theorem 3.2.4. *The price at time t for a zero-coupon bond maturing at time T is*

$$P(t,T) = \exp\{-\int_t^T \varphi(u)du - f_a(T-t)x(t) - f_b(T-t)y(t) + \frac{1}{2}V(t,T)\}.$$

Proof. Since

$$P(t,T) = E\{e^{-\int_t^T r_s ds}|\mathcal{F}_t\},$$

by Lemma 3.2.1, we know that the mean of $\int_t^T r_s ds$ is

$$\int_t^T \varphi(u)du + f_a(T-t)x(t) + f_b(T-t)y(t),$$

and its variance is $V(t,T)$. With the help of Proposition 3.2.3, it is easy to see that our statement is true. □

Corollary 3.2.5. *The stochastic differential equation of $P(t,T)$ is*

$$dP(t,T) = r(t)\cdot P(t,T)dt - f_a(T-t)\sigma P(t,T)dW_1(t) - f_b(T-t)\eta P(t,T)dW_2(t).$$

Proof. This is an easy exercise of Ito's Lemma, since $P(t,T) = h\big(x(t), y(t), t\big)$. In this case, the function f is exponential . We know

$$\begin{aligned} dP(t,T) =& \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial x}dx(t) + \frac{\partial h}{\partial y}dy(t) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}d < x, x > (t) \\ &+ \frac{1}{2}\frac{\partial^2 h}{\partial y^2}d < y, y > (t) + \frac{\partial^2 h}{\partial x \partial y}d < x, y > (t). \end{aligned}$$

From the theorem above, we know

$$\begin{aligned} \frac{\partial h}{\partial x} =& - f_a(T-t)P(t,T); \frac{\partial h}{\partial y} = -f_b(T-t)P(t,T); \\ \frac{\partial^2 h}{\partial x^2} =& f_a^2(T-t)P(t,T); \frac{\partial^2 h}{\partial y^2} = f_b^2(T-t)P(t,T); \\ \frac{\partial^2 h}{\partial x \partial y} =& f_a(T-t)f_b(T-t)P(t,T) \\ \frac{\partial h}{\partial t} =& \bigg(\varphi(t) + x(t)e^{-a(T-t)} + y(t)e^{-b(T-t)} - \frac{\sigma^2}{2a^2}(1-e^{-a(T-t)})^2 - \frac{\eta^2}{2a^2}(1-e^{-b(T-t)})^2 \\ &- \rho\eta\sigma\frac{(1-e^{-a(T-t)})(1-e^{-b(T-t)})}{ab}\bigg) * P(t,T). \end{aligned}$$

So

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} =& \big\{\varphi(t) + x(t)e^{-a(T-t)} + y(t)e^{-b(T-t)} - \frac{\sigma^2}{2a^2}(1-e^{-a(T-t)})^2 \\ &- \frac{\eta^2}{2a^2}(1-e^{-b(T-t)})^2 - \rho\eta\sigma\frac{(1-e^{-a(T-t)})(1-e^{-b(T-t)})}{ab}\big\}dt \\ &- f_a(T-t)\{-ax(t)dt + \sigma dW_1(t)\} - f_b(T-t)\{-by(t)dt \\ &+ \eta dW_2(t)\} + \frac{1}{2}f_a^2(T-t)\sigma^2 dt + \frac{1}{2}f_b^2(T-t)\eta^2 dt \\ &+ f_a(T-t)f_b(T-t)\rho\sigma\eta dt \\ =& \big\{\varphi(t) + x(t) + y(t)\big\}dt - f_a(T-t)\sigma dW_1(t) - f_b(T-t)\eta dW_2(t) \\ =& r(t)dt - f_a(T-t)\sigma dW_1(t) - f_b(T-t)\eta dW_2(t). \end{aligned}$$

□

Chapter 4

Bond Futures

As presented in Chapter 1, a futures contract gives investors an obligation to buy or sell a certain asset at an agreed upon price at a specified future date. Bond futures are futures written on bonds. If the underlying is a bond index, then they are called bond index futures. The traditional bond futures are written on a basket of bonds. At the delivery date, the buyer of bond futures should determine which one is the cheapest-to deliver bond. Currently, there appears a new type of bond futures– total return bond futures. EuroMTS Bond index futures belong to this new type. In this project, we deal with the traditional bond futures with the assumption that the cheapest-to-delivered bond is known. This assumption is also reasonable because the choice of cheapest-to-deliver bonds has little influence on the pricing of the corresponding bond futures. And then we will deal with the total return bond index futures.

4.1 Mathematical Properties of Futures

In order to understand embedded properties of futures contracts and how futures contracts are priced, we give a definition of a futures contract from a mathematical point of view.

Definition 4.1.1. The futures contract on an underlying asset $\mathcal{Y}$ is a financial asset with a price process $\Pi(t)$ and a dividend process $D(t)$ satisfying the following conditions:

$$
\begin{aligned}
&D(t) = F(t; T, \mathcal{Y}) \\
&F(T; T, \mathcal{Y}) = \mathcal{Y} \\
&\Pi(t) = 0, \forall t \leq T
\end{aligned}
$$

where $F(t; T, \mathcal{Y})$ is the futures price for $\mathcal{Y}$ at time t for the delivery at T.

The futures contracts are settled everyday by the clearing house of the exchange. During an arbitrary time interval $(s, t]$ of futures contracts' lifetime, the holder of a contract receives the amount of $F(t; T, \mathcal{Y}) - F(s; T, \mathcal{Y})$. Therefore, the futures contracts can be treated as an asset with dividends every day.

Proposition 4.1.1. *Let $\mathcal{Y}$ be a given contingent T-claim, and assume that market prices are obtained from the fixed risk neutral martingale measure $\mathbb{Q}$. Then the futures prices process is given by*

$$F(t; T, \mathcal{Y}) = E^{\mathbb{Q}}[\mathcal{Y}|\mathcal{F}_t]$$

Proof. Since the futures price can be treated as an asset $\Pi(t)$ with dividends $F(t; T, \mathcal{Y})$, then its discounted gain process is

$$G(t) = \frac{\Pi(t)}{B_t} + \int_0^t \frac{1}{B_s} dF(s; T, \mathcal{Y}).$$

As $\Pi(t)$ mentioned in the definition above is equal to 0, we obtain

$$dF(t; T, \mathcal{Y}) = B_t dG(t).$$

By the following proposition, we know that $G(t)$ is a Q-martingale. Thus, $F(t; T, \mathcal{Y})$ is a $\mathbb{Q}$-martingale as well. By the property of martingale, we would have

$$F(t; T, \mathcal{Y}) = E^{\mathbb{Q}}[F(T; T, \mathcal{Y})|\mathcal{F}_t] = E^{\mathbb{Q}}[\mathcal{Y}|\mathcal{F}_t].$$

□

Proposition 4.1.2 (Björk)**.** *Consider a general factor model satisfying the following assumptions. Assumption 1:The only objects which are a priori given are the following.*

- *An empirically observable k-dimensional stochastic process*

$$X = (X_1, \ldots, X_k),$$

which is not assumed to be the price of a traded asset, with P-dynamics given by

$$dX_i(t) = \mu_i(t, X(t))dt + \delta_i(t, X(t))d\bar{W}(t), i = 1, \ldots, k,$$

where $\bar{W} = (\bar{W}_1, \ldots, \bar{W}_n)$ *is a standard n-dimensional P-Wiener process.*

- *A risk free asset (money account) with the dynamics*

$$dB(t) = rB(t)dt.$$

Assumption 2: The short rate of interest is assumed to be a deterministic function of the factors, i.e.

$$r(t) = f(X(t)).$$

If the market is free of arbitrage, then for any price process S (underlying or derivative) with dividend process D, the normalized gains process

$$Z_t = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_\tau} dD(\tau)$$

is a $\mathbb{Q}$*-martingale.*

For the detailed discussion of this proposition, we refer to Björk(2004). Here, we only use this result.

By Proposition 4.1.1, Theorem 3.2.4 and Proposition 3.2.3, we know that the futures price written on one zero-coupon bond is as formulated in the following theorem.

Theorem 4.1.3. *The futures price* $F(t; T, P)$ *for a future written on a* T^**-zero coupon bond delivered at time* T *(here* $T^* > T$*) is a martingale under the risk neutral measure* $\mathbb{Q}$ *and so*

$$\begin{aligned} F(t; T, P) &= E^{\mathbb{Q}}(F(T; T^*, P)|\mathcal{F}_t) \\ &= E^{\mathbb{Q}}(P(T, T^*)|\mathcal{F}_t) \\ &= e^{M_P(t,T,T^*) + \frac{1}{2}V_P(t,T,T^*)}, \end{aligned}$$

where M_P *and* V_P *are specified below.*

Proof. From Corollary 3.2.5, we know that

$$\begin{aligned} P(T,T^*) &= \frac{P^M(0,T^*)}{P^M(0,T)} exp\{A(T,T^*)\} \\ A(T,T^*) &= \frac{1}{2}[V(T,T^*) - V(0,T^*) + V(0,T)] - f_a(T^*-T)x(T) - f_b(T^*-T)y(T). \end{aligned}$$

The logarithm of $P(T,T^*)$ is normally distributed with the mean

$$\begin{aligned} M_P(t,T,T^*) =& ln\frac{P^M(0,T^*)}{P^M(0,T)} + \frac{1}{2}[V(T,T^*) - V(0,T^*) + V(0,T)] \\ & - f_a(T^*-T)E^{\mathbb{Q}}(x(T)|\mathcal{F}_t) - f_b(T^*-T)E^{\mathbb{Q}}(y(T)|\mathcal{F}_t), \end{aligned}$$

and variance

$$\begin{aligned} V_P(t,T,T^*) =& \mathrm{Var}(f_a(T^*-T)x(T) + f_b(T^*-T)y(T)|\mathcal{F}_t) \\ =& f_a^2(T^*-T)\sigma^2\int_t^T e^{-2a(T-u)}du + f_b^2(T^*-T)\eta^2\int_t^T e^{-2b(T-u)}dW_2(u) \\ & + 2f_a(T^*-T)f_b(T^*-T)\sigma\eta\rho\int_t^T e^{-a(T-u)}e^{-b(T-u)}du \\ =& \sigma^2 f_a^2(T^*-T)f_{2a}(T-t) + \eta^2 f_b^2(T^*-T)f_{2b}(T-t) \\ & + 2f_a(T^*-T)f_b(T^*-T)f_{a+b}(T-t)\sigma\eta\rho. \end{aligned}$$

From Equation 3.1.2, we obtain

$$E^{\mathbb{Q}}\big(x(T)|\mathcal{F}_t\big) = x(t)\cdot e^{-a(T-t)}.$$

Similarly,

$$E^{\mathbb{Q}}\big(y(T)|\mathcal{F}_t\big) = y(t)\cdot e^{-b(T-t)}.$$

Thus the mean of the normally distributed variable of the logarithm of $P(T,T^*)$ turns out to be

$$\begin{aligned} M_P(t,T,T^*) =& \ln\frac{P^M(0,T^*)}{P^M(0,T)} + \frac{1}{2}[V(T,T^*) - V(0,T^*) + V(0,T)] \\ & - f_a(T^*-T)\sigma x(t)\cdot e^{-a(T-t)} - f_b(T^*-T)\eta y(t)\cdot e^{-b(T-t)}. \end{aligned}$$

Then by Proposition 3.2.3, we would know that the price of one bond future is

$$F(t;T,P) = e^{M_P(t,T,T^*)+\frac{1}{2}V_P(t,T,T^*)}.$$

$\square$

The pricing of zero-coupon bonds is essential because a coupon bond $B(t, T)$ can be treated as a linear combination of several discount bonds prices $P(t, T_j)$ in which T_j stands for the time of a coupon payment. Assume bond futures with the delivery time T are written on a bond with maturity $T^* = T_n$, in which $T_j > T$ for $j = 1, \dots, n$. Thus, for a coupon bond $B(t, T_n)$ with par value of 1 and coupon d, we have

$$B(T, T_n) = d \sum_{j=1}^{n} P(T, T_j) + P(T, T_n).$$

By Theorem 4.1.3, we obtain the pricing formula of bond futures $F_b(t, T)$ written on a bond with maturity T_n under the risk neutral measure $\mathbb{Q}$:

$$\begin{aligned} F_b(t, T) &= E^{\mathbb{Q}}(B(T, T^*)|\mathcal{F}_t) \\ &= E^{\mathbb{Q}}(d \sum_{j=1}^{n} P(T, T_j) + P(T, T_n)|\mathcal{F}_t) \\ &= d \sum_{j=1}^{n} E^{\mathbb{Q}}(P(T, T_j)|\mathcal{F}_t) + E^{\mathbb{Q}}(P(T, T_n)|\mathcal{F}_t) \\ &= d \sum_{j=1}^{n} e^{M_P(t,T,T_j) + \frac{1}{2} V_P(t,T,T_j)} + e^{M_P(t,T,T_n) + \frac{1}{2} V_P(t,T,T_n)}. \end{aligned}$$

We have proved the following corollary

Corollary 4.1.4. *The price $F_b(t, T)$ of a bond futures with maturity T written on a coupon bond and coupon dates $t < T_1 < T_2 < \cdots < T_n = T^*$ with par value 1, coupon d and maturity T^* is*

$$F_b(t, T) = d \sum_{j=1}^{n} e^{M_P(t,T,T_j) + \frac{1}{2} V_P(t,T,T)} + e^{M_P(t,T,T_n) + \frac{1}{2} V_P(t,T,T_n)}.$$

4.2 Euribor Futures

The Euribor rates, similar to the LIBOR rates, are the abbreviation of Euro Interbank Offered Rate. They are used as a reference rate for euro-denominated forward rate agreements. We use U to stand for the current quote for a 90-day deposit. Since Euribor rates are simply compounded forward interest rates, one investor who commits N euro today will receive in 90 days the principal and the interest payment of $N \cdot U \cdot (\frac{90}{360})$. With this convention, we will let $U_\delta(t)$ denote the Euribor quote at time t for a deposit of 360δ days. For the case of 90-day deposit, δ is $1/4$.

By the arbitrage argument, the price $P(t,T)$ of one zero-coupon bond at time t and the Euribor rates should satisfy the following relation:

$$P(t, t+\delta)(1 + \delta \cdot U_\delta(t)) = 1,$$

or, equivalently,

$$U_\delta(t) = \frac{1}{\delta}(\frac{1}{P(t, t+\delta)} - 1).$$

From the discussion above, we see that there is a close relation between the Euribor quote and the zero-coupon bond. Our next step is to establish the relation between the Euribor quote and Euribor futures.

Let $F_\delta(t,T)$ be the Euribor futures price at time t written on a deposit commitment of 360δ days with maturity of T. By conventions of the trading exchanges, the final settlement price of the Euribor futures $F_\delta(t)$ with maturity of T should satisfy

$$F_\delta(T,T) = 100(1 - U_\delta(T)).$$

By Proposition 4.1.1, we know that all the futures prices are martingales under the risk-neutral measure $\mathbb{Q}$. We come to the following relation, based on all the discussion above

$$\begin{aligned}
F_\delta(t,T) &= E^{\mathbb{Q}}\big(F_\delta(T,T)|\mathcal{F}_t\big) \\
&= E^{\mathbb{Q}}(100(1-U_\delta(T))|\mathcal{F}_t) \\
&= 100E^{\mathbb{Q}}\big(1-\frac{1}{\delta}(\frac{1}{P(T,T+\delta)}-1)|\mathcal{F}_t\big) \\
&= 100(\frac{1}{\delta}+1)-\frac{100}{\delta}E^{\mathbb{Q}}\big(P(T,T+\delta)^{-1}|\mathcal{F}_t\big)
\end{aligned}$$

By Theorem 3.2.4 and 4.1.3, we easily obtain that

$$F_\delta(t,T) = 100(\frac{1}{\delta}+1)-\frac{100}{\delta}e^{-M_P(t,T,T+\delta)+\frac{1}{2}V_P(t,T,T+\delta)}$$

in which the parameters M_P and V_P are defined as in Theorem 4.1.3.

4.3 Total Return Bonds and Futures

Total return bonds are different from the traditional bonds, which pay coupons to holders, in that generated coupons are reinvested into the bonds right away. With total return bonds, it is easy for investors to manage and compare with other investment opportunities. That is why the total return bonds are getting popular now. The EuroMTS bond index also belongs to this type.

For total return bonds, the prices of the bonds are quoted as clean prices like the traditional bonds, although the coupons are reinvested. When we try to derive the pricing formula of total return bonds, we need to formulate how the amount of bonds changes with respect to the coupon payment dates. The coupons generated by the previously reinvested coupons make it interesting, especially how the futures on total return bonds move. In order to analyze the properties of the EuroMTS bond index, we first study one single total return bond,since the EuroMTS bond index is a basket of several similar bonds with different maturities.

Suppose there is one single total return bond with par value of 1 and coupon payment of d at the times $t_0, t_1, \ldots, t_n = T^*$. The value $B(t, T^*)$ of it at time $t \in [t_i, t_{i+1})$ with maturity T^* is determined by the current clean bond value and the amount of bonds at time t .

At time t_0, we have the first coupon from the bond and reinvest to the bond immediately. Then the current value at time t_0 consists of two parts. The first part is the value of the bond after the coupon payment and the second part is the value of coupon. Here we use $q(t, T^*)$ to express the clean bond price like the traditional bond price quotation, which is the value of the bond after the coupon payment.

$$\begin{aligned} B(t_0, T^*) &= q(t_0, T^*) + \frac{d}{q(t_0, T^*)} q(t_0, T^*) \\ &= (1 + \frac{d}{q(t_0, T^*)}) q(t_0, T^*) \end{aligned}$$

This is the special case we have presented. More generally, for the time $t \in (t_0, t_1)$, the quotation of this total return bond changes to $q(t, T^*)$. Then the value of it becomes

$$B(t, T^*) = (1 + \frac{d}{q(t_0, T^*)}) q(t, T^*)$$

For the next coupon payment time t_1, we consider not only the coupon generated by the total return bond, but also the coupon generated by the previously reinvested coupon. Then the value of the total return bond consists of 4 parts. Assume that the price is $q(t_1, T^*)$ and the value can be expressed as follows:

$$\begin{aligned} B(t_1, T^*) &= q(t_1, T^*) + \frac{d^2}{q(t_0, T^*) * q(t_1, T^*)} q(t_1, T^*) + \frac{d}{q(t_1, T^*)} q(t_1, T^*) + \frac{d}{q(t_0, T^*)} q(t_1, T^*) \\ &= (1 + \frac{d}{q(t_0, T^*)})(1 + \frac{d}{q(t_1, T^*)}) q(t_1, T^*) \end{aligned}$$

This is what happens at time t_1. For the time $t \in (t_1, t_2)$, the value of this total return bond is

$$B(t,T^*) = (1+\frac{d}{q(t_0,T^*)})(1+\frac{d}{q(t_1,T^*)})q(t,T^*)$$

To generalize this process, we conclude that for any time t the value of this total return bond is

$$B(t,T^*) = \prod_{j=1}^{i}(1+\frac{d}{q(t_j,T^*)})q(t,T^*)$$

in which

$$q(t_j,T^*) = d\sum_{k=j+1}^{n} P(t_j,t_k) + P(t_j,T^*)$$
$$q(t,T^*) = d\sum_{k=j+1}^{n} P(t,t_k) + P(t,T^*)$$

We first discuss the futures on one single total return bond. From the Björk proposition, we know that the futures value $F(t,T)$ with the maturity of T at time t would be equal to

$$\begin{aligned} F(t,T) &= E(F(T,T)|\mathcal{F}_t) \\ &= E(B(T,T^*)|\mathcal{F}_t) \end{aligned}$$

Suppose that $T \in [t_m, t_{m+1})$, in which t_m is the coupon payment date of the total return bond. We have

$$B(T,T^*) = \prod_{j=1}^{m}(1+\frac{d}{q(t_j,T^*)})q(T,T^*)$$

Thus, the futures on the total return bonds is as follows:

$$F(t,T) = E(\prod_{j=1}^{m}(1+\frac{d}{q(t_j,T^*)})q(T,T^*)|\mathcal{F}_t)$$

in which $T \in [t_m, t_{m+1})$ and $t \in [t_m, T)$. This is because the standard EuroMTS Government bond index is standardized with a maturity of 3 months. Then at time t, the information on $q(t_j, T^*)$ is known and then the amount of shares reinvested on this bond has been determined as well. The futures pricing formula becomes

$$F(t,T) = \prod_{j=1}^{m}(1 + \frac{d}{q(t_j, T^*)}) \cdot E(q(T, T^*)|\mathcal{F}_t)$$

Then the total return rate bond futures is reduced to the normal bond futures and we can use Corollary 4.1.3 to solve it.

From the discussion above, we see that there is no big difference between total return bond futures and normal bond futures. Actually, the total return bond futures are simpler than normal bond futures, because usually the latter ones incorporates quality options, which complicate their pricing problem. EuroMTS Government Bond Index futures, as a typical total return bond futures, can be analyzed with the help of a single government bond future, because there exist some weights on different government bonds to construct an index. The pricing problem will not be influenced by the weights.

Chapter 5

Calibration of Two-Additive-Factor Gaussian Models

All mathematical models are useless unless they can be applied to practical problems and explain them reasonably. In this chapter, we consider the issue of calibration and survey different calibration approaches. Secondly, we use the real-market data to calibrate the G2++ model.

5.1 What is calibration?

Calibration is the estimation of parameters of a model. It is the very first step to apply the model to the real problem. The basic idea is to calibrate the model to the real data so that we could determine the parameters contained in the model. In our case, we aim to find the estimation of 5 parameters in our G2++ model, which are a ,b, σ, η, and ρ.

Financial institutions may calibrate the same interest rate model in different ways, and generate different prices for the same interest rate product. Certainly, some losses can be caused either by the poor calibration results or by the choice of a calibration method. The first reason may be caused by some errors in the process of data processing. The second reason is fundamental, since it is a system error in the whole calibration process. It is important to investigate the calibration approaches carefully and make a good choice.

5.2 Calibration Approaches

Calibration approaches, which depend on the types of data we use, consist of the historical volatility approach and the implied volatility approach. Historical volatility approach means that volatility parameters are inferred from historical data. And implied volatility approach means that volatility parameters are inferred from current market prices of some financial products. Generally, these two approaches can be applied to all interest rate models, no matter whether they are equilibrium models or term structure consistent models. However, there are some well-known drawbacks on historical volatility approaches, since they can lead to underestimate the volatilities and usually the implied volatilities are higher than the historical volatilities. In contrast, the implied volatility approach allows one to price the financial products with the market expectation. If the benchmark financial products are priced correctly by the incorporation of all the market information, then we can expect that our desired products will also be priced properly.

This calibration method is based on the real measure, which is different from our model. Our G2++ model is based on the risk neutral measure. If we use this calibration method, we may achieve the exact volatility value of the model, because the volatility will not change with respect to the measure. But the drift part of the model will not be the same under different measures. This method will not lead us to a good calibration results.

5.3 Implied Volatilities Approaches

Before implied volatility approaches are used in the calibration, we need to consider several questions.

1. which financial product should be used as the benchmark?

2. which parameters should be time-varying and which should not be?

For the first question, the consensus has been reached by researchers. The benchmark financial products can be chosen based on the following criteria:

1. The benchmark should be liquid, since it is believed to incorporate more market information.

2. The benchmark should be similar to the product to be priced.

In the European fixed income market, the euro swaps and caps are the most heavily traded financial products. For the second question, the consensus has not been reached on this question. The results also depend on the interest rate model. In the G2++ model, we introduce a time-varying function to match the current interest rate curve. We easily avoid the second question.

Within the implied volatility approach, there still exist two different approaches. We will briefly introduce them in the following.

5.3.1 Time-independent Volatility Parameters Approach

If the volatility parameters in a model are constant, we can match the model prices with different maturities to the market prices of the benchmark financial products. To make things specific, we suppose that there are different market prices of euro swaps for different maturities: $P_{1,market},\ P_{2,market},\ldots$. Corresponding, we also can derive the model prices: $P_{1,model},P_{1,model},\ldots$. Then the parameters in the model should be chosen in such a way that the mean square of relative difference between market prices and model prices should be close to zero. That is to say, we should minimize the following function:

$$\sum (modelprices - marketprices)^2.$$

This approach gives a stable estimation of the parameters. If the mean square error is small, we are confident that the estimation of parameters are close to the true market parameters. We will use this method in this project and present our results latter on.

5.3.2 Time-varying Volatility Parameters Approach

Beside time-independent volatility parameters, there exists another method– time-varying volatility parameters approach. Although this method does not suit our case, as the parameters in our model are constant, it is good to mention it to get a general calibration picture.

This approach can be used in a model where all the volatility parameters are functions of time. The volatility parameter has the property of non-stationarity. The problem with this approach is that it might overfit the data. It should be tested whether there is non-stationarity of volatility parameters.

Suppose that we have a model with only one volatility parameter $\sigma(t)$ and the resetting dates of two market caps $P_{1,market}$ and $Q_{1,market}$ are the same, $0 < T_1 < \ldots < T_n$. The volatility parameter $\sigma(t)$ is chosen in such a way that $P_{1,market}$ and $Q_{1,market}$ should be equal within the time interval $(0, T_1)$, (T_1, T_2), and so on. In this way, we can derive a non-stationary volatility parameter $\sigma(t)$ for $t \in [0, T_n]$.

5.4 Empirical Results

The calibration of the two-additive-factor Gaussian model in our case takes the time-independent volatility parameters approach, since the parameters in the model do not vary with respect to time. The introduction of a deterministic function $\varphi(t)$ is supposed to capture the difference between the market term structure of interest rate and the theoretical one.

5.4.1 Description of Data

We take the most liquid fixed income product, Euro swap, as the benchmark to derive the zero coupon curve, since Euro swaps are heavily traded and believed to incorporate more market information into them. The data in Table 5.1 we are using are downloaded from Bloomberg on August 2nd, 2007. For the data in Table 5.1, it is understood as the Euro swap rate is 4.4436 for the maturity of 0.5 year, by taking the example of "Maturity 0.5, Rate 4.4436".

From these euro swap prices, we can derive the zero-coupon curve, which is the essential step in pricing any fixed income product. From the table above, we notice that we only have annual Euro swap rates. From these annual swap rates, we could derive the market discount factors by the bootstrapping method as follows.

Table 5.1 Euro Swap rate on August 2nd, 2007

Maturity(year)	Rate(%)	Maturity(year)	Rate(%)	Maturity(year)	Rate(%)
0.5	4.4436	16	5.05	33	4.9465
1	4.661	17	5.05307	34	4.9415
1.5	4.706	18	5.05609	35	5.00466
2	4.797	19	5.05742	37	4.9285
3	4.84013	20	5.05822	38	4.9235
4	4.8695	21	5.0605	39	4.9185
5	4.891	22	5.0595	40	4.98172
6	4.907	23	5.058	41	4.9085
7	4.922	24	5.055	42	4.9035
8	4.9375	25	5.05	43	4.8985
9	4.956	26	5.048	44	4.8945
10	4.9735	27	5.0435	45	4.9595
11	4.991	28	5.0385	46	4.885
12	5.005	29	5.0335	47	4.8805
13	5.02	30	5.026	48	4.8765
14	5.0305	31	4.9555	49	4.8735
15	5.0415	32	4.9505	50	4.93872

The discount factor for the cash flow calculation period applicable to the first cash flow payment date d1 is defined with respect to the Euro swap rate and the relevant day count fraction for the first cash flow calculation.

$$d_1 = \frac{1}{1 + A_1 * C_1}$$

in which A_1 is the Euro swap rate and C_1 is the relevant day count fraction for this Euro swap rate. For the calculation of the rest of market discount factors, we use the following formula:

$$d_r = \frac{1 - C_r \sum_{i=1}^{r-1} A_i d_i}{1 + A_r C_r}$$

in which

C_r is the Euro swap rate applicable to the r^{th} year;

A_i is the relevant day count fraction for the Euro swap rate applicable to the i^{th} year;

d_i is the discount factor as calculated in accordance with the above formulae, for the Euro swap rate applicable to the i^{th} year.

In our calibration, we need the semi-annual discount rates. Since we have derived the annual discount factors, we use the spline interpolation method to derive the semi-annual discount factors. From all these discount factors, we plot the whole zero-coupon curve in Figure 5.1. The Y-axis is the value of the market discount factor

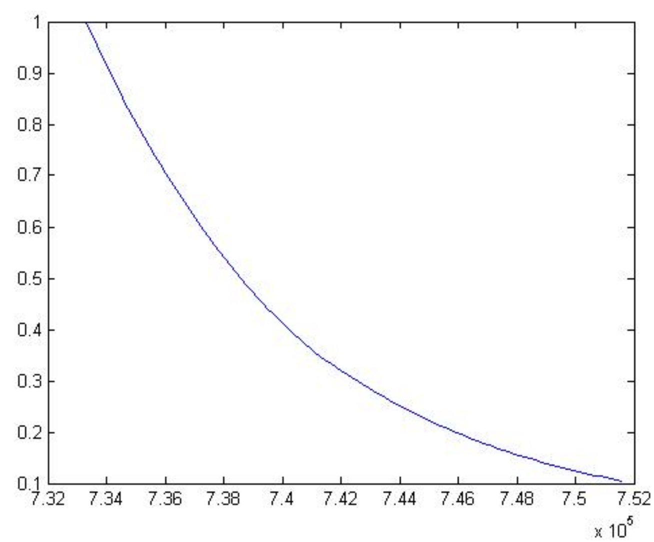

Figure 5.1: The Market Zero Bond Curve

$P(0, T_i)$ and the X-axis of the following curve is the corresponding T_i in the form of serial date numbers instead of usual dates.

For the calibration in next section, we use at-the-money Euro cap-volatility data of August 2nd, 2007 at 4:00 in the afternoon. It is the market standard to price a cap with the following sum of Black's formula. The formula is defined as follows:

$$Cap^{Black}(0, \mathfrak{T}, \tau, K, \sigma_{\alpha,\beta}) = \sum_{i=\alpha+1}^{\beta} P(0, T_i)\tau_i Bl(K, F(0, T_{i-1}, T_i), v_i, 1),$$

where, denoting by ϕ the standard Gaussian cumulative distribution function,

$$\begin{aligned} Bl(K, F, v) &= F\phi(d_1(K, F, v)) - K\phi(d_2(K, F, v)), \\ d_1(K, F, v) &= \frac{ln(F/K) + v^2/2}{v}, \\ d_2(K, F, v) &= \frac{ln(F/K) - v^2/2}{v}, \\ v_i &= \sigma_{\alpha,\beta}\sqrt{T_{i-1}}. \end{aligned}$$

The notations in the formula above is define as follows: $P(0, T_i)$ is the market discount factor at time T_i , K is the strike rate of Euro cap, $F(0, T_{i-1})$ is the market forward rate between time T_{i-1} and T_i, τ_i is the day count fraction between T_{i-1} and T_i, $\sigma_{\alpha,\beta}$ is the common volatility parameter retrieved from market quotes, and α and β are the time interval of the Euro cap rate. Based on the formula above, we find the market prices of Euro cap prices by using the market volatility of Euro caps. The Euro cap rates corresponding to the quotes of the market volatility are shown in Table 5.2.

Table 5.2 Market Euro Cap volatility and prices

Maturity(year)	Market Volatility	Market Cap Rate
1	0.0695	0.00079
2	0.1092	0.00091
3	0.1271	0.001054
4	0.1336	0.001117
5	0.1368	0.001176
6	0.1385	0.001281
7	0.1391	0.001482
8	0.139	0.001868
9	0.1384	0.002377
10	0.1377	0.003268
11	0.1261	0.012541
12	0.1244	0.014307

In the next section, we will explain how we calibrate the G2++ model to the market data by using the date above.

5.4.2 Calibration

As specified in Section 5.3.1, the estimation of our parameters is performed by minimizing our objective function, the sum of the squares of the difference between market cap prices and model cap prices. The latter can be obtained by the closed-form cap pricing formulas of the G2++ model.

There are several constraints on our parameters, which are specified in Chapter 3. We incorporate these constraints on 5 parameters into our objective function by the introduction of the absolute function and the arctan function. Then the field of these 5 parameters will be in the real numbers. The optimization tool in Matlab we use the $fminsearch$ function. The detailed description of this function can be found in any version of Matlab. In this project, we define the maximal function evaluations to be 10000 and the maximal iterations to be 10000 as well. Although the function $fminsearch$ might not be the fastest one, it is very stable to find the minimum.

In order to speed up the optimization, we need to start with a good guess point. This point is found by the comparison of the value of the objective function derived in the process of $fminsearch$ optimization. We start from -1 to 1 with a step of 0.1.

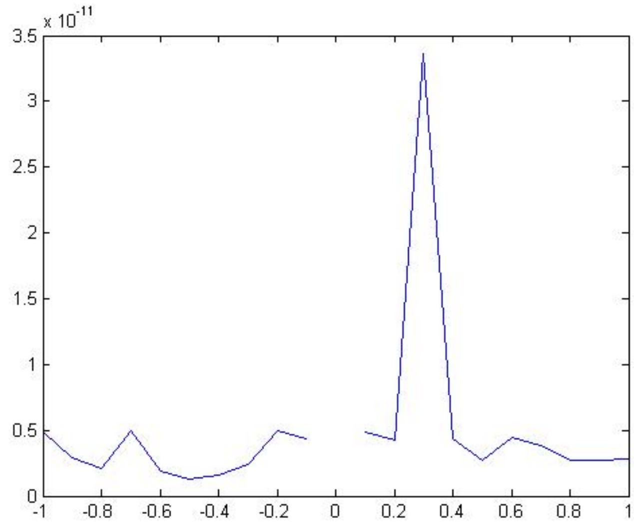

Figure 5.2: The Values of Minimization

With the initial guess range [-1,1], we achieve Figure 5.2, in which the X-axis is the value of the initial guess and the Y-axis is the value of the objective function. We notice that the minimal objective function value is achieved at -0.5 with the value of the objective function of $1.3044e-012$ in our case. The corresponding parameter vector is $[0.97996, 0.10998, 0.027275, 0.0099788, -0.63264]$. Because our initial guesses in the optimization above are the same value, it is possible that the optimization can be improved with the different initial guess such as $[0.9, 0.8, 0.7, 0.6, 0.8]$. Then we use the vector $[0.97996, 0.10998, 0.027275, 0.0099788, -0.63264]$ as the initial value. And we come to the following parameters:$a = 1.6682$, $b = 0.10253$, $\sigma = 0.053668$, $\eta = 0.0093683$, $\rho = -1$ with the value of our objective function of $9.7975e - 013$. This indeed improves our minimization.

By looking at the estimated parameters, we notice that the correlation ρ is -1, which has also been explained by Brigo(2006). "It often happens that the ρ value is quite close to minus one, which implies that the G2++ model tends to degenerate into a one-factor short rate process. The reason is due to our choice of Euro caps, because caps price do not depend on the correlation of forward rates." We believe that the G2++ model has an advantage over one-factor models, but the different data should be used to calibrate the model to avoid getting perfect correlation. One of the financial products which depend on the correlation of forward rates is Euro swaption. The calibration with respect to Euro swaption is deserved for the future work.

Although the method we use above gives us a good result, we still doubt whether the optimization method is reasonable. With regard to this question, we do the following test.We use $fminsearch$ twice to optimize the objective function. For the first optimization, we start with the same initial guess and then redo this process with the parameters generated by the first optimization. The algorithm we use here is as follows:

```
counter=1
for i = -1:0.1:1
InitialGuess=[i i i i i]
terparameters error1= fminsearch( Objective, InitialGuess)
parameters error2= fminsearch( Objective, interparameters)
```

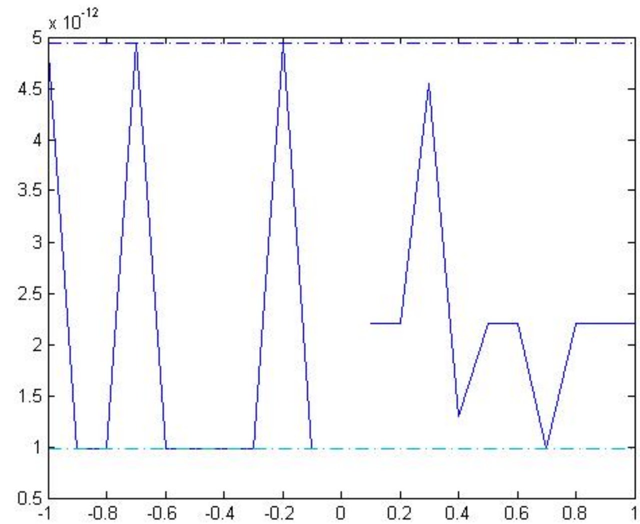

Figure 5.3: Convergence of Minimization

```
error(counter,1)=i

error(counter,2)=error2

end
```

We plot the vector of *error* in Figure 5.3, where the X-axis is the value of the initial guess and the Y-axis is the value of the objective function. We notice that the value of the objective function indeed converges to $9.7975e - 013$.

Although we start with the same initial guesses, our method always leads us to the same minimum of the objective function in the figure above, which proves that our method is trustable.

Comparing the difference between market cap prices and model cap prices based on our parameters, we find that the relative differences between them are small. The fitting quality of the G2++ model is high, which can be seen from the table below, since the relative difference between model cap prices and market cap prices is rather low.

Maturity	Market Rate	Model Rate	Relative difference
1	0.00079034	0.00079034	-5.6793e-014
2	0.00091221	0.00090966	0.0027953
3	0.0010537	0.0010535	0.0001763
4	0.0011176	0.0011174	0.00024665
5	0.0011767	0.0011763	0.00035446
6	0.0012809	0.0012809	-3.9115e-005
7	0.0014817	0.001482	-0.00021973
8	0.0018676	0.0018679	-0.00014474
9	0.0023767	0.0023765	8.8368e-005
10	0.0032686	0.0032683	9.5751e-005
11	0.012541	0.012541	2.9352e-005
12	0.014307	0.014307	-3.3967e-005

Chapter 6

Hedging

In finance, a hedge is an investment that is implemented to reduce or cancel out the risk in another investment. Hedging is a strategy that is used to minimize the exposure to an undesired risk, while still profiting from its business activity. In this project, we use two different interest rate futures to design a hedging strategy. In order to do so, we first test the theoretical hedging and then provide the empirical hedging results based on it.

6.1 Theoretical Hedging

Strategies involving futures can be described as either hedging strategies or speculative strategies. In the case of shares portfolio, by using the futures, we can lock in the value of the current shares portfolio. And we also can reduce the undesired trading risk.

Assume that we hold a share of portfolio of S&P 500 index. In order to lock in the current value of this portfolio, we sell an index futures and hold the contract to the maturity of the futures contract. If the index falls after the sell of the index futures, we have a loss on our portfolio but a gain on the index futures. If the index rises, we have a gain on the portfolio and a loss on the index futures. In this way, we guarantee the value of our shares portfolio.

This examples is about the security market. But this hedging idea can also be applied to the fixed income market. In this project, we are desired to do the same kind of hedging, which means to construct a self-financing portfolio with different interest rate futures. For a self-financing portfolio, we must have the following dynamics in continuous time

$$dV(t) = \sum_{i=1}^{N} h_i(t) dS_i(t); \tag{6.1.1}$$

$$V(t) = \sum_{i=1}^{N} h_i(t) S_i(t), \tag{6.1.2}$$

in which h_i is the weight of the asset S_i in the portfolio and V is the total value of this portfolio.

If we hedge one bond futures, for an example, with Euribor futures of 4 different maturities, then $V(t)$ would be the value of this bond future and $S_i(t)$ will be the value of one Euribor futures with the corresponding weight h_i in the portfolio. Now we start to derive the precise hedging strategies. In order to do it, we need to have stochastic differential equations for Euribor futures and bond futures.

We start with some intermediate calculations. As

$$\begin{aligned} M_P = & \ln \frac{P^M(0,T^*)}{P^M(0,T)} + \frac{1}{2}[V(T,T^*) - V(0,T^*) + V(0,T)] \\ & - f_a(T^* - T)\sigma \int_0^t e^{-a(T-u)} dW_1(u) - f_b(T^* - T)\eta \int_0^t e^{-b(T-u)} dW_2(u), \end{aligned}$$

in order to ease the notation, we introduce $B_{T^*} = -\sigma f_a(T^* - T)$, $C_{T^*} = -\eta f_b(T^* - T)$ and take $R(t) = \int_0^t e^{-a(T-u)} dW_1(u)$, $S(t) = \int_0^t e^{-b(T-u)} dW_2(u)$. Then the SDE form of $R(t)$ and $S(t)$ are

$$dR(t) = e^{-a(T-t)}dW_1(t),$$

$$dS(t) = e^{-b(T-t)}dW_2(t).$$

So now suppose $g\big(R,S\big) = e^{B_{T^*}R(t)+C_{T^*}S(t)}$.Thus, we have by the Ito's formula

$$\begin{aligned}
g_R(R,S) &= B_{T^*} \cdot g(R,S);\\
g_S(R,S) &= C_{T^*} \cdot g(R,S);\\
g_{RR}(R,S) &= B_{T^*}^2 \cdot g(R,S);\\
g_{SS}(R,S) &= C_{T^*}^2 \cdot g(R,S);\\
g_{RS}(R,S) &= B_{T^*} \cdot C_{T^*} \cdot g(R,S)\\
g_t(R,S) &= 0.
\end{aligned}$$

$$\begin{aligned}
dg\big(R,S\big) =& d\{e^{B_{T^*}R(t)+C_{T^*}S(t)}\}\\
=& B_{T^*} \cdot g(R,S) \cdot dR(t) + C_{T^*} \cdot g(R,S) \cdot dS(t) + \frac{1}{2}B_{T^*}^2 \cdot g \cdot d < R > (t)\\
&+ \frac{1}{2}C_{T^*}^2 \cdot g(R,S) \cdot d < S > (t) + B_{T^*} \cdot C_{T^*} \cdot g(R,S) \cdot d < R,S > (t)\\
=& B_{T^*} \cdot g(R,S)\big(e^{-a(T-t)}dW_1(t)\big) + C_{T^*} \cdot g(R,S)\big(e^{-b(T-t)}dW_2(t)\big)\\
&+ \frac{1}{2}B_{T^*}^2 \cdot g(R,S)\sigma^2 e^{2a(T-t)}dt + \frac{1}{2}C_{T^*}^2 \cdot g(R,S)\eta^2 e^{2b(T-t)}dt\\
&+ B_{T^*} \cdot C_{T^*} \cdot g(R,S)\sigma\eta\rho e^{-(a+b)(T-t)}dt\\
=& \Big\{\frac{1}{2}B_{T^*}^2 \cdot g(R,S)\sigma^2 e^{2a(T-t)} + \frac{1}{2}C_{T^*}^2 \cdot g(R,S)\eta^2 e^{2b(T-t)}\\
&+ B_{T^*} \cdot C_{T^*} \cdot g(R,S)\sigma\eta\rho e^{-(a+b)(T-t)}\Big\}dt + B_{T^*} \cdot g(R,S)e^{-a(T-t)}dW_1(t)\\
&+ C_{T^*} \cdot g(R,S)e^{-b(T-t)}dW_2(t)\\
=& U(T^*)dt + B_{T^*} \cdot g(R,S)e^{-a(T-t)}dW_1(t) + C_{T^*} \cdot g(R,S)e^{-b(T-t)}dW_2(t),
\end{aligned}$$

in which

$$\begin{aligned} U(T^*) =& \frac{1}{2} B_{T^*}^2 \cdot g(R,S)\sigma^2 e^{2a(T-t)} + \frac{1}{2} C_{T^*}^2 \cdot g(R,S)\eta^2 e^{2b(T-t)} \\ & + B_{T^*} \cdot C_{T^*} \cdot g(R,S)\sigma\eta\rho e^{-(a+b)(T-t)}. \end{aligned}$$

In Section 4.2, we have shown that the price of one Euribor futures with maturity T at time t is

$$F_\delta(t,T) = 100(\frac{1}{\delta}+1) - \frac{100}{\delta} e^{-M_{P(T,T+\delta)}+\frac{1}{2}V_{P(T,T+\delta)}}.$$

We have

$$\begin{aligned} dF_\delta(t,T) &= d\{100(\frac{1}{\delta}+1) - \frac{100}{\delta} e^{-M_{P(T,T+\delta)}+\frac{1}{2}V_{P(T,T+\delta)}}\} \\ &= -\frac{100}{\delta} d\{e^{-M_{P(T,T+\delta)}+\frac{1}{2}V_{P(T,T+\delta)}}\}. \end{aligned}$$

Given that δ, T and T^* are fixed, the term $V_{P(T,T+\delta)}$ is deterministic. We can factor out the variance part. To ease our equation, we take

$$l(T,T+\delta) = e^{\frac{1}{2} * V_{P(T,T+\delta)}}.$$

Thus we apply the Ito's Lemma and obtain the following

$$\begin{aligned} dF_\delta(t,T) =& -\frac{100}{\delta} e^{\frac{1}{2}V_{P(T,T+\delta)}} d\{e^{-M_{P(T,T+\delta)}}\} \\ =& -\frac{100}{\delta} l(T,T+\delta) \Big\{ U(T+\delta)dt - B_{T+\delta} \cdot g(R,S) e^{-a(T-t)} dW_1(t) \\ & - C_{T+\delta} \cdot g(R,S) e^{-b(T-t)} dW_2(t) \Big\} \\ =& -\frac{100}{\delta} l(T,T+\delta) U(T+\delta)dt + \frac{100}{\delta} l(T,T+\delta) B_{T+\delta} \cdot g(R,S) e^{-a(T-t)} dW_1(t) \\ & + \frac{100}{\delta} l(T,T+\delta) C_{T+\delta} \cdot g(R,S) e^{-b(T-t)} dW_2(t), \end{aligned}$$

in which

$$\begin{aligned} U(T+\delta) =& \frac{1}{2} B_{T+\delta}^2 \cdot g(R,S)\sigma^2 e^{2a(T-t)} + \frac{1}{2} C_{T+\delta}^2 \cdot g(R,S)\eta^2 e^{2b(T-t)} \\ & + B_{T+\delta} \cdot C_{T+\delta} \cdot g(R,S)\sigma\eta\rho e^{-(a+b)(T-t)}. \end{aligned}$$

For the price $F_b(t,T)$ at time t of one bond futures with maturity T, the delivered bond for this futures is with maturity T^* and the coupon rate p. From the previous section and the Ito's Lemma, we obtain that

$$
\begin{aligned}
dF_b(t,T) =& p\sum_{j=1}^{n} d\Big\{e^{M_{P(T,T_j)}+\frac{1}{2}V_{P(T,T_j)}}\Big\} + d\Big\{e^{M_{P(T,T_n)}+\frac{1}{2}V_{P(T,T_n)}}\Big\} \\
=& p\sum_{j=1}^{n} e^{\frac{1}{2}V_{P(T,T_j)}} d\Big\{e^{M_{P(T,T_j)}}\Big\} + e^{\frac{1}{2}V_{P(T,T_n)}}\Big\{e^{M_{P(T,T_n)}}\Big\} \\
=& p\sum_{j=1}^{n} l(T,T_j)\Bigg\{U(T_j)dt + B_{T_j}\cdot g(R,S)e^{-a(T-t)}dW_1(t) + C_{T_j} \\
& \cdot g(R,S)e^{-b(T-t)}dW_2(t)\Bigg\} + l(T,T_n)\Bigg\{U(T_n)dt + B_{T_n} \\
& \cdot g(R,S)e^{-a(T-t)}dW_1(t) + C_{T_n}\cdot g(R,S)e^{-b(T-t)}dW_2(t)\Bigg\} \\
=& \Bigg\{p\sum_{j=1}^{n} l(T,T_j)U(T_j) + l(T,T_n)U(T_n)\Bigg\}dt \\
& + \Bigg\{p\sum_{j=1}^{n} l(T,T_j)B_{T_j}\cdot g(R,S)e^{-a(T-t)} + l(T,T_n)B_{T_n} \\
& \cdot g(R,S)e^{-a(T-t)}\Bigg\}dW_1(t) + \Bigg\{p\sum_{j=1}^{n} l(T,T_j)C_{T_j}\cdot g(R,S)e^{-b(T-t)} \\
& + l(T,T_n)C_{T_n}\cdot g(R,S)e^{-b(T-t)}\Bigg\}dW_2(t).
\end{aligned}
$$

As we are going to hedge this bond futures with Euribor futures, we need 4 Euribor futures with different maturities: T_1, T_2, T_3 and T_4 to capture the stochastic process of the bond future. The self-financing portfolios are constructed as follows:

$$
\begin{aligned}
dF_b(t,T) &= \sum_{i=1}^{4} h_i dF_\delta(t,T_i); \\
V(t) &= \sum_{i=1}^{4} h_i(t)S_i(t).
\end{aligned}
$$

From all the discussion above, we would achieve the following

$$p\sum_{j=1}^{n} l(T,T_j)U(T_j) + l(T,T_n)U(T_n) = -\frac{100}{\delta}\sum_{i=1}^{4} h_i \cdot l(T_i,T_i+\delta)U(T_i+\delta);$$

$$\begin{aligned} p\sum_{j=1}^{n} l(T,T_j)B_{T_j}\cdot g(R,S)e^{-a(T-t)} &+ 100l(T,T_n)B_{T_n}\cdot g(R,S)e^{-a(T-t)} \\ &= \frac{100}{\delta}\sum_{i=1}^{4} h_i \cdot l(T_i,T_i+\delta)B_{T_i+\delta}\cdot g(R,S)e^{-a(T_i-t)}; \end{aligned}$$

$$\begin{aligned} p\sum_{j=1}^{n} l(T,T_j)C_{T_j}\cdot g(R,S)e^{-b(T-t)} &+ 100l(T,T_n)C_{T_n}\cdot g(R,S)e^{-b(T-t)} \\ &= \frac{100}{\delta}\sum_{i=1}^{4} h_i \cdot l(T_i,T_i+\delta)C_{T_i+\delta}\cdot g(R,S)e^{-b(T_i-t)}, \end{aligned}$$

$$V(t) = \sum_{i=1}^{4} h_i(t)S_i(t)$$

The derivation above elaborates the exact idea of how the hedging strategies can be designed with bond futures and Euribor futures. Certainly, there is no difference in the design of hedging strategies between total return bond futures and Euribor futures.

6.2 An Example

In this section, we present how we could derive the hedging weights in the practice according to our equations above. We will use all the calibration results and pricing formulas. In the previous chapter, we have achieved the values of all the parameters in the G2++ model. All the pricing formulas in Chapter 3 and 4 are not symbolic any more.

6.2.1 Paths of Brownian Motions

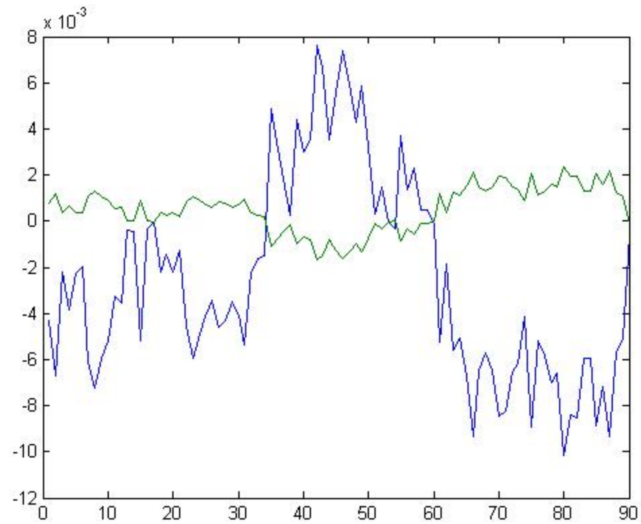

Figure 6.1: Simulated path of two correlated Brownian Motions

As all the pricing formulas consist of two Brownian motions, we need to generate these dynamics first. In this case, our stochastic dynamics are based on two correlated Brownian motions. Due to the Gaussian properties of Brownian motions, we are able to discretize the pathes of Brownian motions. Take the process $x(t)$ as an example. To apply a numerical method to it, we first discretize the interval [s,t]. Let $\delta t = (t-s)/N$ for some positive integer N. Then the stochastic process $x(t)$ can be written as

$$x(t) \approx x(s)e^{-a(t-s)} + \sigma \sum_{i=1}^{N} e^{-a(t-t_i)}(W_{t_{i+1}} - W_{t_i})$$

Similarly, the process $y(t)$ can be written as

$$y(t) \approx y(s)e^{-b(t-s)} + \eta \sum_{i=1}^{N} e^{-b(t-t_i)}(W_{t_{i+1}} - W_{t_i})$$

In Matlab, these processes can be easily generated. One simulation is shown in Figure 6.1, in which the X-axis is the maturity in 90 days and the Y-axis is the value of the

Euribor future. From Section 4.2, we know that Euribor futures are a function of

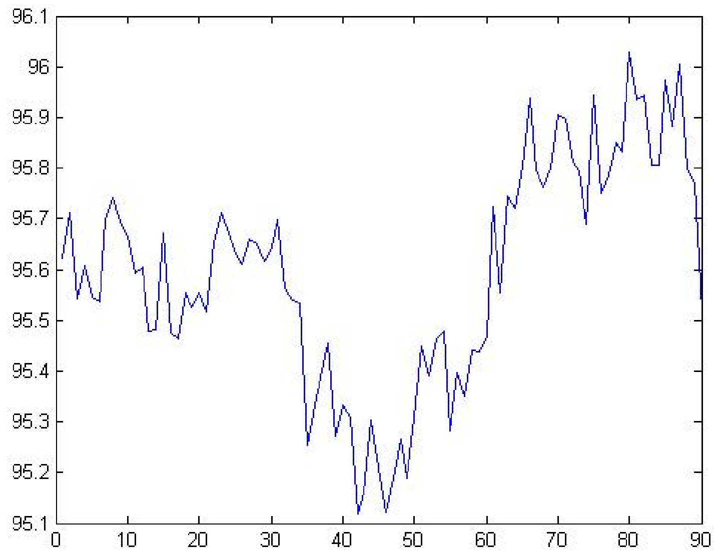

Figure 6.2: Simulated path of Euribor Futures

$x(t)$ and $y(t)$. From the simulation above,the corresponding paths of Euribor futures can also be derived.

Figure 6.2 and 6.3 are the paths of Euribor futures and zero-coupon bonds prices based on one single simulation, respectively. In Figure 6.2, the X-axis stands for the 90-day maturity of one Euribor future and the Y-axis is the value of the Euribor future. We notice that the generated path of Euribor futures is rather reasonable with respect to the current real interest rate level, which is around 4.5%, since the range of Euribor futures should be between 95 and 96. In Figure 6.3, the X-axis stands for the maturity of futures in 90 days and the Y-axis is the value of zero coupon bonds. The generated zero coupon curve is not a smooth curve. This might be due to the Monte Carlo method, as we would like to incorporate some stochastics.

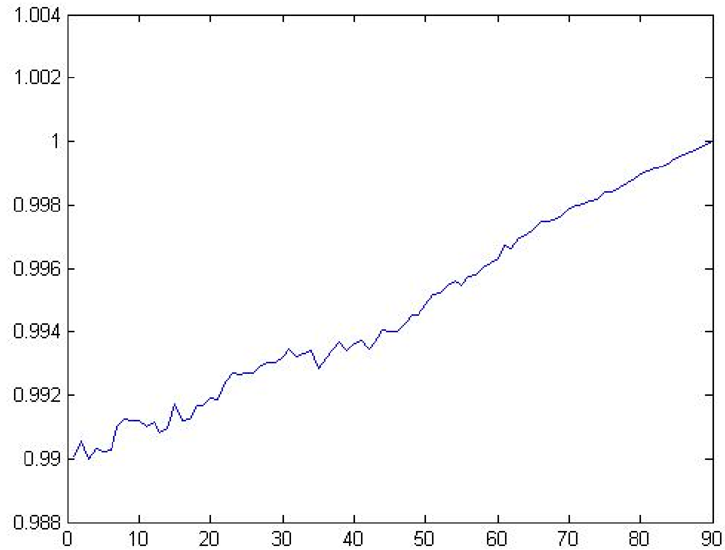

Figure 6.3: Simulated path of Zero Coupon Curve

6.2.2 Generation of Futures Paths

The goal of this project is to test the possibility of hedging EuroMTS Government Bond index futures with Euribor futures. As we have discussed in Section 4.3, EuroMTS Government Bond index is a collection of total return bonds, which has the properties of one single total return bond. The futures on EuroMTS Government Bond index can also be analyzed by the futures on the total return bond. In this subsection, we present our theoretical hedging results regarding to this issue.

We suppose that there has been issued one total return bond with a maturity of 7 years and the annual coupon rate of 3% on June 23rd, 2007. The par value of this bond is 1. The future on this total return bond is written today with the maturity of 3 months. Thus, the pricing formula of the future can be derived easily based on the discussion in Section 4.3.

$$F(t,T) = E(q(T,T^*)|\mathcal{F}_t)$$

The generated path of this bond future is as follows: in which the X-axis is the

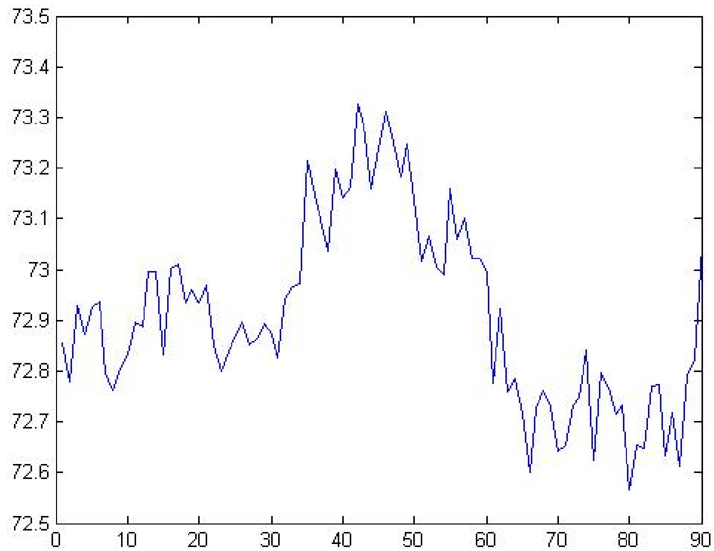

Figure 6.4: The Generated Bond Futures

maturity of 90 days for one bond future and the Y-axis is the value of the bond future. This seems to be a special case, since we write a future on the new issued bond. But there is no big difference when the futures are written on the bond issued a few years ago.

$$F(t,T) = \prod_{j=1}^{m}(1 + \frac{d}{q(t_j,T^*)}) \cdot E(q(T,T^*)|\mathcal{F}_t)$$

As we have discussed in Section 4.3, the term $\prod_{j=1}^{m}(1 + \frac{d}{q(t_j,T^*)})$ is a constant by the time t. The problem returns to the normal bond futures.

For the total bond futures, we are planning to hedge it with 4 Euribor futures with the maturity of 1/2 year, 1/4 year, 1 year, and 2 years, respectively. From the paths of $x(t)$ and $y(t)$ in Figure 6.1, we generate the paths of these 4 Euribor futures, which is shown in Figure 6.5. The X-axis of the figure is the maturity of 90 days and the Y-axis is the value of futures.

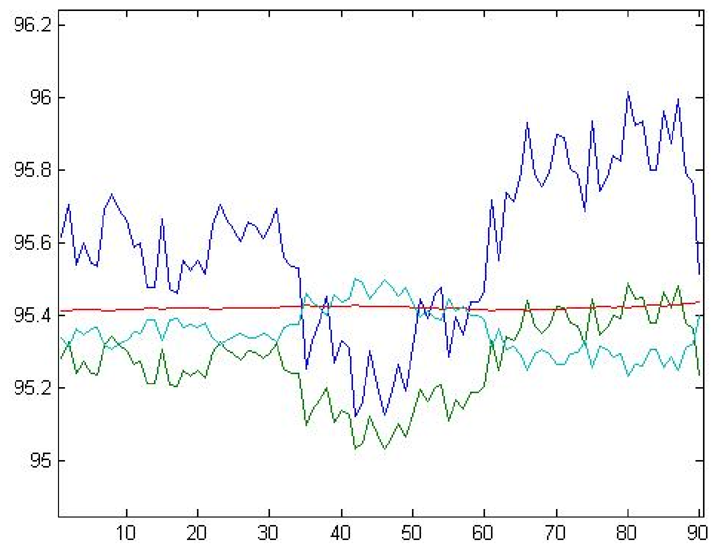

Figure 6.5: The Generated pathes of 4 Euribor Futures

The hedging weights on these 4 different Euribor futures can be found in Appendix A. To plot the hedging portfolio and the bond future together, we have Figure 6.6 and 6.7, in which the X-axis is also the maturity of 90 days and the Y-axis is the value of futures (difference in Figure 6.7).

Chapter 7

Conclusions

In this project, we adopt the two-additive-factor Gaussian model to derive the pricing formulas of interest rate futures, as we are interested in hedging EuroMTS Government Bond Index futures with Euribor futures and the precise capture of the term structure of the market interest rate is crucial. Due to the Gaussian properties of it, many closed-form pricing formulas of financial products have been derived so far. We choose the Euro cap prices as our benchmark and calibrate the G2++ model to the real market data.

Our calibration confirms the results on the correlation parameter in Brigo(2006). Due to the choice of Euro caps, the correlation coefficient in our calibration is minus one. This is due to the property of Euro caps, since Euro caps do not depend on the forward rates. Therefore, in many situations, one-factor models can fit the cap data well. If we use Swaption as our calibration benchmark, we will achieve the different parameters with respect to it. The correlation coefficient might be significantly different from -1. This requires some further research on this issue. But the G2++ model is certainly better than its one-factor counterpart, since the non-Markovian property makes the G2++ model outstanding.

With our calibration results, we could replicate the Euribor futures path, which appears to be reasonable according to the current interest rate level. Besides, we could also generate the paths of all the futures, which include Euribor futures, total return bond futures. Certainly, we need to generate the stochastic dynamics for them.

To put the hedging idea into practice, we need to test all the results by the real market data, based on our theoretical derivation. There are several cruxes in the practice. Firstly, since EuroMTS Government Bond index are rebalanced every month, the weights on individuals bonds will definitely change. The selected bonds in the index will change every month. How these factors influence the hedging strategies should definitely be considered. Secondly, our pricing formula on total return bond futures is only based on one single bond. As EuroMTS Government Bond Index consists of a basket of bonds, the pricing formula may be more complex than the current one. So The more general pricing formula in the G2++ model should be studied to incorporate all the market facts. The more interesting question is how this hedging strategy between EuroMTS Government Bond index futures and Euribor futures perform under the real market data. This definitely deserves more research effort.

Appendix A

The Hedging Results

In this appendix, we present all the hedging weights on 4 Euribor futures with different maturities. Here, EF1 stands for the weights on the Euribor futures with the maturity of 1/4 year, EF2 for the weights on Euribor futures with the maturity of 1/2 year, EF3 for the weights on Euribor futures with the maturity of 1 year, and EF4 for the weights on Euribor futures with the maturity of 2 years.

Time index(day)	EF1	EF2	EF3	EF4
1	8.5594	-13.2006	5.4817	-0.1139
2	8.4162	-12.959	5.3794	-0.1117
3	8.3531	-12.8414	5.3286	-0.1106
4	8.2202	-12.6166	5.2337	-0.1086
5	8.1253	-12.4507	5.1632	-0.1071
6	8.0164	-12.2636	5.0841	-0.1054
7	7.8598	-12.0042	4.9754	-0.1031
8	7.7385	-11.7993	4.8894	-0.1013
9	7.644	-11.6357	4.8205	-0.0998
10	7.545	-11.4654	4.749	-0.0982
11	7.4592	-11.3157	4.6861	-0.0969
12	7.3506	-11.1318	4.6093	-0.0953
13	7.2793	-11.0047	4.5559	-0.0941

Time index(day)	EF1	EF2	EF3	EF4
14	7.1753	-10.8285	4.4826	-0.0926
15	7.0246	-10.5826	4.381	-0.0904
16	6.9725	-10.4853	4.3402	-0.0895
17	6.875	-10.3203	4.2719	-0.0881
18	6.7538	-10.1201	4.1895	-0.0863
19	6.6629	-9.9659	4.1259	-0.085
20	6.5583	-9.7916	4.0544	-0.0835
21	6.4713	-9.6438	3.9937	-0.0822
22	6.3452	-9.4385	3.91	-0.0805
23	6.2383	-9.2622	3.8381	-0.0789
24	6.1554	-9.122	3.781	-0.0777
25	6.0708	-8.9795	3.723	-0.0765
26	5.9866	-8.838	3.6657	-0.0753
27	5.8863	-8.6734	3.5992	-0.0739
28	5.8001	-8.5298	3.5412	-0.0727
29	5.7195	-8.3947	3.4869	-0.0716
30	5.6273	-8.2432	3.4261	-0.0703
31	5.531	-8.0863	3.3633	-0.069
32	5.4728	-7.985	3.3228	-0.0681
33	5.3935	-7.8536	3.2704	-0.067
34	5.3114	-7.7184	3.2167	-0.0659
35	5.2819	-7.6594	3.1933	-0.0654
36	5.1852	-7.5039	3.1318	-0.0641
37	5.0941	-7.357	3.0739	-0.0629
38	5.0019	-7.209	3.0157	-0.0617
39	4.9559	-7.1273	2.9838	-0.0611
40	4.8668	-6.9847	2.9279	-0.0599
41	4.7937	-6.8651	2.8813	-0.059
42	4.7492	-6.7864	2.8509	-0.0583

Time index(day)	EF1	EF2	EF3	EF4
43	4.6663	-6.6538	2.7995	-0.0573
44	4.5679	-6.4998	2.7398	-0.056
45	4.5096	-6.4026	2.7025	-0.0553
46	4.45	-6.3041	2.6648	-0.0545
47	4.3674	-6.1736	2.6147	-0.0535
48	4.2847	-6.0437	2.5651	-0.0524
49	4.226	-5.9474	2.5286	-0.0517
50	4.1387	-5.8119	2.4771	-0.0506
51	4.0507	-5.6761	2.4256	-0.0496
52	3.992	-5.581	2.3899	-0.0489
53	3.9159	-5.4625	2.3454	-0.048
54	3.8487	-5.3567	2.3059	-0.0472
55	3.8105	-5.2907	2.2819	-0.0467
56	3.7318	-5.17	2.2369	-0.0457
57	3.6751	-5.0795	2.2037	-0.0451
58	3.6016	-4.9669	2.1621	-0.0442
59	3.5408	-4.8717	2.1273	-0.0435
60	3.4767	-4.7726	2.0911	-0.0428
61	3.3875	-4.6403	2.0425	-0.0418
62	3.3493	-4.5767	2.0201	-0.0414
63	3.2702	-4.4589	1.9772	-0.0405
64	3.2168	-4.3758	1.9476	-0.0399
65	3.1522	-4.2782	1.9127	-0.0392
66	3.0833	-4.1755	1.876	-0.0385
67	3.045	-4.1133	1.8546	-0.038
68	2.9956	-4.037	1.828	-0.0375
69	2.9393	-3.9519	1.7983	-0.0369
70	2.8773	-3.86	1.7661	-0.0363
71	2.8276	-3.7844	1.7401	-0.0357
72	2.7861	-3.7198	1.7183	-0.0353
73	2.7384	-3.6474	1.6937	-0.0348

Time index(day)	EF1	EF2	EF3	EF4
74	2.6995	-3.5866	1.6735	-0.0344
75	2.6282	-3.4851	1.6384	-0.0337
76	2.5987	-3.4369	1.6231	-0.0334
77	2.5489	-3.3634	1.5985	-0.0329
78	2.4975	-3.2883	1.5734	-0.0324
79	2.454	-3.2235	1.5523	-0.032
80	2.3935	-3.1379	1.5236	-0.0314
81	2.3577	-3.0835	1.5064	-0.0311
82	2.3141	-3.0196	1.4859	-0.0307
83	2.2826	-2.9711	1.471	-0.0304
84	2.2408	-2.9101	1.4518	-0.03
85	2.1871	-2.8352	1.4276	-0.0296
86	2.1536	-2.7851	1.4124	-0.0293
87	2.1049	-2.717	1.3909	-0.0288
88	2.0804	-2.6784	1.38	-0.0286
89	2.0439	-2.6257	1.3641	-0.0283
90	2.0233	-2.592	1.3552	-0.0281

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