Modeling Liquidity Risk on Interest Products

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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science

Business Analytics
Faculty of Exact Sciences

August 2013
VRIJE UNIVERSITEIT AMSTERDAM

Abstract

Faculty of Exact Sciences
Business Analytics
Master of Science

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In the recent financial crisis large corporations and governments have experienced the
effects of drying up money markets: premiums must be paid to attract the required
funding, such that the total costs run significantly higher than would be expected
from the prevailing interest rate. Currently there are no models known to predict and
quantify this liquidity premium in money-markets, which consist mostly of interest
products.

In this thesis a proper way to model the described liquidity premiums on such products
is sought. A distinction between short and medium-term funding requirements is made,
as both cases are thought to have different effects on the term structure.

Hull-White one and two factor models combined with a liquidity model by Bertsimas
and Lo form the basis for the short term models, of which two variants are recommended.
Direct simulations of the interest-product price using the same liquidity model and a
smoothing method is recommended for medium-term problems.

A recursive learning procedure is proposed that can be followed to calibrate the Hull-
White based models on live scenarios. The learning procedure can also be applied on
the medium-term scenarios, albeit slightly more computationally intensive.

Future studies in which the models are applied in practice are required to determine
the final best choice model. This is because the conditions in which the models can be
applied are rare, and cannot be observed historically either because they do not occur
in isolated form. Until this empirical model validation has been applied two models are
recommended for both the short- as well as the medium-term scenarios.
Acknowledgements

First, I want to express my gratitude to my supervisors at Ernst & Young, Diederik Fokkema and Anton Bossenbroek. Especially in the first months of this research Anton’s suggestions and critical thinking were of great value to me. Diederik’s critical reading and remarks have ensured that this thesis is written succinctly wherever possible. Without their continued support this thesis would not have reached its current state. Secondly my primary supervisor at the VU Amsterdam, Rob van der Mei. The supportive conversations we had once a month kept me motivated and on track. Also I would like to thank my secondary university contact person André Ran, for finding the time to read my thesis thoroughly on such short notice. And of course thanks to all my colleagues for the friendly work environment and the many interesting lunch meetings we had.

Finally I would like to thank my friends and family, for keeping me motivated throughout the harder times of this internship.
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Section 1

Introduction

In the financial crisis of 2007 we have observed large companies, banks and even governments in serious financial distress because they were not able to attract funding at the regular rate. Different causes for this problem exist, such as the money market drying up, decreased investor trust, decrease in creditworthiness of a party, and many more. In this thesis we will investigate problems that occur when a party is trying to attract a large sum of money in an illiquid market. We will consider the problems on a large scale, proportional to the need of a country’s government. Often such requirements go along with an increased funding price due to a lack of creditworthiness, such as was the case with Greece. We will consider the scenarios where creditworthiness is not up for question, but problems arise solely due to market illiquidity.

We consider a pre-crisis situation: The market is still actively trading, but there clearly is a downward trend and investors are less easily lending their money. This is a global, or at least European-wide situation, which is not expected to change any time soon. Now, in this economic climate, an otherwise financially healthy country such as the Netherlands is in an immediate, extreme need for liquidity: Within five business days they require the amount of EUR 100,000,000. Note that the actual amount does not matter, although it must in this context be large enough to disrupt the funding market. Depending on the market conditions that can be at any amount. In the pre-order situation, the Dutch government is expected to pay the current interest level for their funding requirement, say 3.5%. However, once the order for the funding comes through short interest rates spike to 5%, increasing the financial impact for the Netherlands by EUR 1,500,000. Now, the actual numbers are fictional, but the problem should be clear: Could the government have predicted and quantified the spiking interest rate? If so, could they also have formulated a strategy which would have minimized the overall impact?
In the above scenario we assume that the financial requirements of a single party can influence the prevailing interest rate significantly. It is unlikely that smaller companies can have this effect, but for countries or large banks in an economically declining climate this may definitely be true. In a normal situation, the interest level that should be paid for a loan with a given maturity can be extracted from the yield curve of zero coupon bonds. Figure 1.1 displays the normal yield curve in a fictional situation. In this research we follow the idea that this curve in fact consists of two components: a ‘normal’ component, and a liquidity premium. Normally this liquidity premium is negligible, but in scenarios such as the one sketched above the second component becomes significant, possibly leading to unexpected, enormous losses. The task for the Dutch government in the above scenario would be to attract the funds within the required time frame, while minimizing the weighted yield-curve shifts due to the liquidity premium.

The problem sketched above has one important feature that we have not yet considered: the time frame for which the party requires funding. There may be a funding gap for several years, this is typically a problem with deeper roots. There are also the shorter-term problems, where a liquidity mismatch persists only for weeks or several months. Both situations call for a different modeling approach, as we will see in the sequel.

![Figure 1.1: Illustration of liquidity premium on yield curve.](image)

Note that here the increasing interest rate is due to a lack of available funds, i.e. a liquidity problem. In situations where a party’s creditworthiness is questioned the shift is caused by multiple drivers. Modeling creditworthiness and the problems that arise is a topic that is already being researched actively, and combining the situations is something we recommend for further research. The interested reader can consider reading up on credit-value adjustments to get familiar with this topic.
Section 2

Problem Definition

In the introduction we have sketched the scenario that we will follow throughout this research: an otherwise financially healthy country with an immediate need for funds in an illiquid money market. Funds can be attracted from the money market through different instruments, we will be studying money attractions through different forms of interest-rate products such as bonds and commercial paper. To properly approach this problem the following central question has been composed:

*How can we relate, model and quantify liquidity risk on interest-rate products?*

We split this central question into five smaller ones, which we will follow throughout this research:

1. What are the properties of different liquidity risk models we can find in literature?
2. What are the properties of different interest-rate models we can find in literature?
3. How can existing liquidity risk models be modified such that they can be applied to interest-rate products?
4. How do yield curves behave in financially distressed situations?
5. How can the realized liquidity model be calibrated on market data such that it produces results we would expect when compared to reality?

This thesis is structured as follows. We will start by studying liquidity models that have been researched in recent history in section 3. We will then briefly consider several interest models that are commonly used in practice in section 4. Then, in section 5, we will propose multiple different approaches through which liquidity on interest products can be modeled. Before we can analyze the model output we first set our expectations,
by studying behavior of the yield curve in section 6. In section 7 we will discuss the output of our implementations, and in the final sections we present our conclusions and recommendations.

Once we have answered the above questions, and implemented a model that best captures the relation between interest-rate behavior and liquidity, we may continue towards calibration and testing. Note that modifying and combining models comes with many degrees of freedom: since this is a relatively untouched field there are little right or wrong approaches. We will consider different variants of models, and attempt to merge them without removing too much of either’s properties. We will see later that quantitative validation of the results is not a trivial task, mostly due to amount of mathematical freedom. We will therefore present our findings on a relative basis, rather than with absolute performance figures.
Section 3

Liquidity Models

In this section we will discuss different liquidity models that are proposed in recent literature. Different takes on modeling liquidity exist, but we see one clear distinction: models where liquidity is modeled directly, and models where liquidity is a result of other processes. We will consider a set of models by Bertsimas and Lo, which is of the latter category. Before doing so we will first discuss Hamilton Jacobi Bellman equations, which are fundamental in the derivations of their model. In section 3.3 we will discuss a model that follows the direct approach, where liquidity is modeled using multiplicative error measurement (MEM).

3.1 Hamilton Jacobi Bellman equations

The model by Bertsimas and Lo (B&L) depends heavily on stochastic equations known as the Hamilton Jacobi Bellman (HJB) equations. For a better understanding of the derivations that follow in subsequent sections, we will briefly discuss HJB equations in this section.

HJB equations typically arise in optimal control problems. Optimal control problems are stochastic problems of the following form: there is a process \( x(\cdot) \), whose dynamics are formulated by stochastic differential equation(s) (SDEs). The process \( x(\cdot) \) can be influenced by actions \( \alpha(\cdot) \in \mathcal{A} \) through the differential equation(s). Different actions lead to different behavior of \( x(\cdot) \). Now, let there be a performance indicator \( V(\cdot) \), dependent on the process \( x(\cdot) \). The optimal control problem is to choose 'the right actions at the right times', as to optimize the performance indicator over a (possibly infinite) time horizon.
Before we can solve the above set of problems and derive the HJB equation, we first introduce some notation. First, let the dynamics of \( x(\cdot) \) be defined as follows:

\[
dx(s) = b(x(s), s, \alpha(s, x(s))) + \sigma(x(s), s, \alpha(s, x(s))) dW(s) \tag{3.1}
\]

where \( b(\cdot), \sigma(\cdot) \) continuous functions on \( \mathbb{R} \)

\( x(s) \in \mathcal{X} \) the statespace

\( \alpha(s, x(s)) \in \mathcal{A} \) the actionspace

\( s \in [0, T] \)

Equation 3.1 is not the most general formulation of the state dynamics used in the HJB framework. In the above formulation the optimal action depends only on the current state, which results in controls known as Markov controls. One of the advantages of this group of controls is that an optimal policy always exists and can be found. [1]

Second, the performance indicator can be bounded or unbounded in both the time direction and the state direction. Throughout, we will focus solely on problems that are bounded in both directions, i.e. a fixed time horizon and bounded state space. The performance indicator, or value function, is written (in its general form) as follows:

\[
J(x, t, \alpha(\cdot)) = \mathbb{E} \left[ \int_t^T f(x(s), s, \alpha(s)) ds + g(x(T), T) \mid \mathcal{F}(x, t) \right] \tag{3.2}
\]

In equation 3.2 running costs \( f(\cdot) \) and final costs \( g(\cdot) \) together form the value function \( J(\cdot) \). The expectation is taken at each decision moment, such that the latest state information is used. Hence the filtration on \( x \) and \( t \).

Now, the objective is to find a set of controls \( \alpha^*(\cdot) \), such that 3.3 holds:

\[
J(x_0, 0, \alpha^*(\cdot)) = \min_{\alpha(\cdot) \in \mathcal{A}} J(x_0, 0, \alpha(\cdot)) \tag{3.3}
\]

To this end, we first write the optimal costs to go at time \( t \) as follows:

\[
u(x, t) = \min_{\alpha(\cdot) \in \mathcal{A}} J(x, t, \alpha(\cdot)) \tag{3.4}\]

Now, from Bellman's principle of optimality we may write \( u(x, t) \) in terms of \( u(x, t + dt) \) as follows:

\[
u(x, t) = \min_{\alpha(\cdot) \in \mathcal{A}} f(x(t), t, \alpha(\cdot)) dt + u(x(t + dt), t + dt) \tag{3.5}\]
Applying Taylor expansion, canceling $u(x,t)$ on both sides, ignoring terms of $o(dt)$ and dividing by $dt$ yields:

$$u(x,t) = \min_{\alpha(\cdot) \in A} f(x(t), t, \alpha(\cdot))dt + u(x(t), t)dt + \frac{d}{dx} u(x(t), t)x(t)dt + o(dt)$$

$$\min_{\alpha(\cdot) \in A} f(x(t), t, \alpha(\cdot))dt + u(x(t), t)dt + \frac{d}{dx} u(x(t), t)x(t)dt + o(dt) = 0 \quad (3.6)$$

$$\lim_{dt \to 0} \frac{\cdot}{dt} = u(x, t) + \min_{\alpha(\cdot) \in A} \left[ \frac{d}{dx} u(x, t)x(t) + f(x, t, \alpha(\cdot)) \right] = 0 \quad (3.7)$$

This is the HJB equation in its simplest form. Under the assumptions we have made this far (Markov property, boundedness) it is a necessary and sufficient condition for optimality, and as such yields the optimal solution the optimal control problem.

HJB equations generally do not have closed-form solutions, although for some specific value functions closed-form solutions have been derived. Examples of closed-form solutions are the minmax framework in game theory, and viscosity solutions. Since closed-form solutions are generally lacking, the HJB equation needs to be solved via backward recursion, using the fact that the value at the boundary $t = T$ is known, and thus $u(x, T)$ is known. We will see both backward recursion and closed-form solutions in the next section, where we will discuss the liquidity model by Bertsimas and Lo.

### 3.2 A liquidity-equity model by Bertsimas and Lo

Liquidity problems are closely related to the problem of buying or selling a large portion of stock at minimal execution cost. This can be understood easily: suppose the model returns optimal execution strategies for buying a portion of stock at minimal cost. We can transform that model to selling a number of shares at maximum profit, by simply telling it to buy a negative number of shares (some model manipulations are probably necessary). Now, selling a number of shares at maximum profit $X$ is closely related to the problem of realizing profit $Y$ by selling as little shares as necessary. The last scenario is our liquidity problem, where very high numbers of shares need to be sold to meet financial obligations.

Bertsimas and Lo [2] propose an algorithm that minimizes executing costs for buying orders. They choose a stock as the underlying product and propose different methods to model execution costs. Although we are interested in liquidity problems related to interest products, their model does provide a good starting point. B&L propose three slightly different algorithms with closed-form solutions, followed by a more general
Recursive approach. In the following subsections we will have a close look at their derivations.

3.2.1 The Basic Model

Suppose a trader wishes to buy an amount of stock $\bar{S}$ within time frame $[0, T]$. We define $P_t$ as the mean price paid to obtain $S_t$ shares at time $t$. If we assume the market is perfectly liquid and elastic, then the trader could simply place a buying order for $\bar{S}$ shares at $T_0$, and obtain the shares for the total price $P_0 \bar{S}$, $P_0$ being the current price of the stock.

However, since we are studying liquidity problems we wish to drop the assumption of a perfectly liquid market. The trader now faces the challenge to place his buying orders such that the extra execution costs are minimal. If we discretize $[0, T]$ into $T$ time steps the mathematical representation is as follows:

$$
\min \mathbb{E} \left[ \sum_{t=1}^{T} P_t S_t \right] \\
s.t. \sum_{t=1}^{T} S_t = \bar{S}
$$

Bertsimas and Lo now propose multiple laws of motion that describe the behavior of $P_t$. Of course, the choice for a particular law of motion is crucial for the optimal execution costs. For some laws of motion closed-form expressions for the optimal execution strategy can be found, while for more complex laws numerical approximation is required. The most basic law of motion they propose is the following:

$$
P_t = P_{t-1} + \theta S_t + \varepsilon_t, \quad \theta > 0, \quad \mathbb{E}[\varepsilon_t | S_t, P_{t-1}] = 0, t = 1, 2, \ldots, T \tag{3.9}
$$

Here, $\theta$ is the permanent effect of the purchases of the trader, and $\varepsilon$ is random white noise from the market. From the above equation we can clearly see that all price impacts from buying shares are permanent, i.e. there is no market-resilience component. This basic model does not capture the price dynamics we see in the actual market, but it serves as a starting point from which B&L build towards more complex and realistic laws of motion that are (more) able to do so. The solution to the above equation can therefore function as a benchmark for other execution strategies. In the derivation below we will see that the optimal solution to equation $3.9$ is the naive strategy $S_t = \bar{S}/T$.

This entails spreading the buy order evenly across all available trading moments. To retrieve the optimal strategy to equation $3.9$ we first introduce the remaining number of shares to be bought. At $t = 0$ this equals $\bar{S}$, at $t = T + 1$ all shares must be obtained.
We write:

\[ W_t = W_{t-1} - S_{t-1}, \quad W_1 = S, \quad W_{T+1} = 0, \quad t = 1, 2, \ldots, T \]  

(3.10)

We can now write the optimal value function at time \( t \) as an HJB equation:

\[ V_t(P_{t-1}, W_t) = \min_{S_t} \mathbb{E} \left[ P_t S_t + V_{t+1}(P_t, W_{t+1}) \mid \mathcal{F}(P_{t-1}, W_t) \right], \quad t = 1, 2, \ldots, T \]  

(3.11)

We may now use backward recursion starting at \( t = T \) and derive the optimal execution strategy. At time \( T \) the recursion is simple:

\[ V_T(P_{T-1}, W_T) = \min_{S_T} \mathbb{E} \left[ P_T W_T \mid \mathcal{F}(P_{T-1}, W_T) \right] = (P_{T-1} + \theta W_T)W_T \]  

(3.12)

\[ S_T^* = W_T \]  

(3.13)

Next, we express \( V_{T-1} \) in terms of \( S_{T-1}, W_{T-1} \) and \( P_{T-2} \):

\[ V_{T-1}(P_{T-2}, W_{T-1}) = \min_{S_{T-1}} \mathbb{E} \left[ P_{T-1} S_{T-1} + V_T(P_{T-1}, W_T) \mid \mathcal{F}(P_{T-2}, W_{T-1}) \right] \]  

(3.14)

\[ = \min_{S_{T-1}} \mathbb{E} \left[ (P_{T-2} + \theta S_{T-1} + \epsilon_{T-1})S_{T-1} \right. \]  

\[ + V_T(P_{T-2} + \theta S_{T-1} + \epsilon_{T-1}, W_{T-1} - S_{T-1}) \mid \mathcal{F}(P_{T-2}, W_{T-1}) \]  

\[ = \min_{S_{T-1}} \mathbb{E} \left[ (P_{T-2} + \theta S_{T-1} + \epsilon_{T-1})S_{T-1} + \left( P_{T-2} \right. \right. \]  

\[ + \theta S_{T-1} + \epsilon_{T-1} + \theta(W_{T-1} - S_{T-1}) \left) (W_{T-1} - S_{T-1}) \mid \mathcal{F}(P_{T-2}, W_{T-1}) \right] \]  

(3.16)

After grouping terms and taking \( \mathbb{E}[\epsilon_t] = 0 \) we minimize w.r.t \( S_{T-1} \):

\[ V_{T-1}(P_{T-2}, W_{T-1}) = \min_{S_{T-1}} P_{T-2} * W_{T-1} + \theta S_{T-1}^2 - \theta S_{T-1} W_{T-1} + \theta W_{T-1}^2 \]  

(3.17)

\[ \frac{d}{dS_{T-1}} = \theta(2S_{T-1} - W_{T-1}) \]  

(3.18)

\[ S_{T-1}^* = W_{T-1}/2 \]  

(3.19)

\[ V_{T-1}(P_{T-2}, W_{T-1}) = W_{T-1}(P_{T-2} + \frac{3}{4}\theta W_{T-1}) \]  

(3.20)
We repeat this procedure \( T - 1 \) times to find \( V_1 \):

\[
V_1(P_0, W_1) = \mathbb{E}_1 \left[ \sum_{t=1}^{T} P_t S_t^* \right] = P_0 \bar{S} + \frac{\theta \bar{S}^2}{2} \left( 1 + \frac{1}{T} \right)
\]

\[
S_1^* = S_2^* = \cdots = S_T^* = \frac{\bar{S}}{T}
\]

Thus, if the price of the underlying equity follows the simple control law stated above, the optimal execution strategy is simply to spread the buy order equally over the number of available time slots. B&L name this strategy "the naive strategy", as it is the optimal solution to the most naive liquidity model. Its simplicity is the direct result of a control law with only one factor that influences price permanently. Since the execution cost function now becomes a convex function of \( S_t \), its optimal solution is where the marginal execution costs are equal, i.e. the order is split equally across all time steps.

### 3.2.2 Extension: Linear Price impact with information

To make the behavior of the model more realistic, Bertsimas and Lo propose to include an additional serially correlated state variable \( X_t \) that affects price linearly. This state variable should be thought of as the general consensus of the direction of the market. The new price function is as follows:

\[
P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t, \quad \theta > 0, \quad t = 1, 2, \ldots, T
\]

\[
X_t = \rho X_{t-1} + \eta_t, \quad \rho \in (-1, 1)
\]

Where \( \{\eta_t\}, \ t = \{1, 2, \ldots, T\} \) is another white noise process with its own variance \( \sigma^2_\eta \). This additional variance has significant influence on the optimal execution policy, which
can now be formulated as follows (see Appendix C for a full derivation):

\[ S^*_T-k = \delta_{w,k}W_{T-k} + \delta_{x,k}X_{T-k}, \quad k = 0, 1, \ldots, T - 1 \]  

(3.25)

\[ V_{T-k}(P_{T-k-1}, X_{T-k}, W_{T-k}) = \]

\[ P_{T-k-1}W_{T-k} + a_k W_{T-k}^2 + b_k X_{T-k} W_{T-k} + c_k X_{T-k}^2 + d_k \]

(3.26)

where \[ \delta_{w,k} = \frac{1}{k+1} \]  

(3.27)

\[ \delta_{x,k} = \frac{\rho \beta_{k-1}}{2a_{k-1}} \]  

(3.28)

\[ a_k = \frac{\theta}{2} \left(1 + \frac{1}{k+1}\right) \]  

(3.29)

\[ b_k = \gamma + \frac{\theta \rho \beta_{k-1}}{2a_{k-1}} \]  

(3.30)

\[ c_k = \rho^2 c_{k-1} - \frac{\rho^2 \beta_{k-1}^2}{4a_{k-1}} \]  

(3.31)

\[ d_k = d_{k-1} + c_{k-1} \sigma^2_\eta \]  

(3.32)

\[ a_0 = \theta, \quad b_0 = \gamma, \quad c_0 = 0, \quad d_0 = 0 \]  

(3.33)

Note that subscripts \( w, x \) are name indicators, unlike subscript \( k \).

The above model has some advantages over the model we have discussed above, but as noted by B&L, still has severe limitations that don’t stroke with reality. The main issues that still need to be resolved before the model can be of practical value are the following:

- Price effects are linear, whereas in reality they are more likely proportional.
- Price effects are permanent, whereas reality suggests both a permanent and temporary component in price effects.
- Negative prices have positive probabilities, whereas in reality this never happens for stocks.

B&L provide some numerical examples in their paper. We have replicated their input, with the main purpose of verifying our model. Although the model has a large random component to it, we may say that our outcomes are mostly similar to theirs, and thus that our implementation is correct. We will use this common sense verification in subsequent sections as well, but numeric verification will also be done in those cases where Bertsimas and Lo provide more data. Figure 3.1 has been generated by taking the following variables: \( \bar{S} = 100,000, \quad P_0 = 50, \quad T = 20, \quad \theta = 5 \times 10^{-5}, \quad \gamma = 5 \quad \sigma^2_\eta = 0.001, \quad \sigma^2_e = 0.125^2 \) and \( \rho = -0.5 \).
3.2.3 Extension: Linear-Percentage Temporary Price Impact

To overcome the three limitations mentioned above, B&L propose one more extension before continuing towards a completely general formulation. The first step is to split the current price into two components:

\[ P_t = \tilde{P}_t + \Delta_t, \quad t = 1, 2, \ldots, T \quad (3.34) \]

where

- \( P_t \) the price at time \( t \)
- \( \tilde{P}_t \) the underlying price at time \( t \)
- \( \Delta_t \) the temporary price effect at time \( t \)

Now, to overcome the negative price problem, the behavior of \( \tilde{P} \) is modeled as geometric Brownian motion. The temporary price impact \( \Delta_t \) is taken as the current permanent price \( \tilde{P}_t \) multiplied by the effect of share purchases \( \theta S_t \) and serially correlated market information \( X_t \). This yields the following three relations:

\[
\begin{align*}
\tilde{P}_t &= \tilde{P}_{t-1} \exp(Z_t), \quad Z \sim N(\mu_z, \sigma_z), \quad t = 1, 2, \ldots, T \\
\Delta_t &= (\theta S_t + \gamma X_t) \tilde{P}_t \\
X_t &= \rho X_{t-1} + \eta_t
\end{align*}
\]

\[ (3.35) \quad (3.36) \quad (3.37) \]

Note that the Brownian motion ensures that \( \tilde{P}_t \) is non-negative, but does not necessarily guarantee that \( P_t \) is non-negative as well. Mild restrictions on \( \Delta_t \) can be set such that it does apply to \( P_t \) too [2].

**Figure 3.1:** Realization of best execution strategy under linear price impact model assumptions.
This third law of motion in equations 3.35–3.37 still has a closed-form solution, albeit more complex than the ones we have seen before. For a full derivation the reader should refer to Appendix C, the derivation concludes with the following closed-form solution:

\[
S_{T-k}^* = \delta_{x,k}X_{T-k} + \delta_{w,k}W_{T-k} + \delta_{1,k}, \quad k = 0, 2, \ldots, T-1 \tag{3.38}
\]

\[
V_{T-k}(\tilde{P}_{T-k-1}, X_{T-k}, W_{T-k}) = q\tilde{P}_{T-k-1} \left( a_k + b_k X_{T-k} + c_k X^2_{T-k} + d_k X_{T-k} W_{T-k} + e_k W_{T-k} + f_k W^2_{T-k} \right)
\]

where

\[
q = \exp \left( \mu \frac{\sigma^2}{2} \right) \tag{3.40}
\]

\[
\delta_{x,k} = \frac{q\rho d_{k-1} - \gamma}{2(\theta + qf_{k-1})}, \quad \delta_{w,k} = \frac{qf_{k-1}}{\theta + qf_{k-1}}, \quad \delta_{1,k} = \frac{qe_{k-1} - 1}{2(\theta + qf_{k-1})} \tag{3.41}
\]

\[
a_k = \delta_{1,k}(1 + \theta \delta_{1,k}) + q(a_{k-1} + \sigma^2 c_{k-1}) - q\delta_{1,k}(e_{k-1} - \delta_{1,k}f_{k-1}) \tag{3.42}
\]

\[
b_k = q\rho b_{k-1} - \delta_{x,k}(qe_{k-1} - 1) \tag{3.43}
\]

\[
c_k = \delta_{x,k}(\theta \delta_{x,k} + \gamma) + q\rho^2 c_{k-1} - q\delta_{x,k}(\rho d_{k-1} - \delta_{x,k}f_{k-1}) \tag{3.44}
\]

\[
d_k = \gamma \delta_{w,k} + q\rho d_{k-1}(1 - \delta_{w,k}) \tag{3.45}
\]

\[
e_k = \delta_{w,k} + q(1 - \delta_{w,k})e_{k-1} \tag{3.46}
\]

\[
f_k = \theta \delta_{w,k} \tag{3.47}
\]

Note that subscripts \(w, x, 1\) are name indicators, unlike subscript \(k\).

Again, we verify our model implementation by comparing the results to those of Bertsimas and Lo. Figure 3.2 displays a realization of the best execution strategy. These results are plausible, but because of the strong random component there is no proper way to definitively conclude that our model produces similar results just by considering sample execution strategies.

![Sample Optimal execution Strategy under Linear Percentage Price impact](image-url)

**Figure 3.2:** Realization of Optimal execution Strategy under linear percentage price model assumptions.

We can compare the results of our implementation to those of B&L more closely by
completely redoing their Monte Carlo simulation. The parameters they used in their simulation are as follows: $\bar{S} = 100.000$, $P_0 = 50$, $T = 20$, $\theta = 5 \times 10^{-7}$, $\mu_z = 0$, $\sigma^2_z = 0.02^2 / 13$, and $\sigma^2_\eta = 1 - \rho^2$. $\gamma$ and $\rho$ are as given in table 3.1. The table is structured as follows: each $(\gamma, \rho)$-combination lists the expected extra costs per share (in cents) incurred under the optimal strategy, the mean extra costs extra costs per share over 50,000 simulations and the standard deviations extra costs in the simulations. These values are listed for our implementation, and for B&L their implementation as well.

When we compare the results we see that the $S^*$ and $S^*(B&L)$ columns closely match. This means that our implementations have the same expectation, as well as mean realization over 50,000 simulations. Furthermore, the variance of the results is nearly identical. The deviations that we still see can be attributed to the number of simulations, which at only 50,000 is expected to yield some variance. The observed results imply that our implementation is correct, and thus we can later use it for manipulations. If we would not have been able to replicate B&L’s results we could not have drawn any conclusions from the models we plan to develop.

### 3.2.4 Generalization

Bertsimas and Lo conclude with a more general approach to minimizing expected execution costs. This approach generally does not have closed-form solutions, but since it is a dynamic programming problem it can still be solved recursively as long as the problem is well posed. The target function as well as the constraint remain unchanged, but the formulation of the law of motion is far more general. Note that the laws of
motion we have seen before fit into the framework below as well.

\[ \min_{S_t} E \left[ \sum_{t=1}^{T} P_t S_t \right] \]  
(3.48)

\[ \text{s.t.} \quad \sum_{t=1}^{T} S_t = \bar{S} \]  
(3.49)
\[ P_t = f_t(P_{t-1}, X_t, S_t, \varepsilon_t) \]  
(3.50)
\[ X_t = g_t(X_{t-1}, \eta_t) \]  
(3.51)
\[ W_t = W_{t-1} - S_{t-1}, \quad W_1 = \bar{S}, \quad W_{T+1} = 0 \]  
(3.52)

We still have an expression for \( V_{T-k} \), and once the minimization in 3.48 is done we have also found the new value for \( S^*_{T-k} \). B&L define the function \( h(\cdot) \) as below to clarify that \( S^*_{T-k} \) depends solely on \( P_{T-k-1}, X_{T-k}, \) and \( W_{T-k} \).

\[ V_{T-k} (P_{T-k-1}, X_{T-k}, W_{T-k}) = \]  
\[ \min_{S_{T-k}} E_{T-k} \left[ f_{T-k}(P_{T-k-1}, X_{T-k}, S_{T-k}, \varepsilon_{T-k}) S_{T-k} \right. \]  
\[ \left. + V_{T-k+1}(f_{T-k}(\cdot), g_{T-k+1}(\cdot), W_{T-k} - S_{T-k}) \right] \]  
\[ S^*_{T-k} = h_{T-k}(P_{T-k-1}, X_{T-k}, W_{T-k}) \]  
(3.54)
\[ k = 0, 1, \ldots, T - 1 \]

Although the extension we have seen in the previous section is able to take away most immediate modeling objections from a market perspective, this last recursive approach does lend itself far better for adjustments. We want to modify liquidity such that the underlying is an interest-rate product rather than equity, for which we will need to make adjustments. Interest-rate product follow different pricing rules, therefore it is unlikely that we will be able to model it with the same control laws suggested in [2].

Again, we want to verify our implementation. We may do so by plugging in laws of motion similar to the ones we have used before, and compare the result of 50,000 simulations to the results of B&L their simulation. We start by plugging in the most basic law of motion, without any serially correlated state variable and permanent price impact. We can easily verify whether the resulting strategy is correct: it should be the naive strategy \( \bar{S}/T \).

Table 3.2 displays the number of shares to be bought at different times, and different price indexes. The remaining number of shares is fixed at 10. The price space has been discretized into 80 segments, all equally spaced. The state space is irrelevant in this problem as it does not influence price under the simple model assumptions. If
Table 3.2: Optimal execution strategy under simple model, state and remaining shares fixed.

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P=1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>$P=40$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>$P=60$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>$P=80$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

the price is in the lower half of the discretization space we observe a naive strategy $\bar{S}/T$. However, if the price increases towards the upper boundary of the price space the strategy becomes to hold off purchases in expectation of lower prices. This is a problem because in reality there is no upper limit on price. This problem can be mitigated only by choosing the state space large enough, and only consider the lower half of the observed strategies.

Similarly we want to observe how well our implementation replicates the results of the "Linear price impact with information" model. In their simulation Bertsimas and Lo use $\rho = 0.5$ and $\theta = 5 \times 10^{-5}$. Other variables remain the same. We have an additional value, block size, to consider. It is not feasible to use a block size of 1 share in the recursion, which we would ideally want. Block size 1000 is feasible and yields acceptable results.

Table 3.3: Recursively formulated Linear price impact with information model compared to Bertsimas and Lo data

<table>
<thead>
<tr>
<th>$t$</th>
<th>$P_t$</th>
<th>$S_t^*$</th>
<th>$\gamma X_t$</th>
<th>$V_t/1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50</td>
<td>5081.25</td>
<td>-0.079</td>
<td>5111.25</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>5081.25</td>
<td>-0.158</td>
<td>5111.25</td>
</tr>
<tr>
<td>3</td>
<td>49.875</td>
<td>5037.50</td>
<td>-0.316</td>
<td>5061.25</td>
</tr>
<tr>
<td>4</td>
<td>49.75</td>
<td>5011.75</td>
<td>-0.079</td>
<td>5061.25</td>
</tr>
<tr>
<td>5</td>
<td>49.75</td>
<td>5041.00</td>
<td>0.079</td>
<td>5061.25</td>
</tr>
<tr>
<td>6</td>
<td>49.875</td>
<td>5039.88</td>
<td>0</td>
<td>4961.75</td>
</tr>
<tr>
<td>7</td>
<td>50</td>
<td>4994.13</td>
<td>0</td>
<td>4911.88</td>
</tr>
<tr>
<td>8</td>
<td>50</td>
<td>4956.13</td>
<td>-0.158</td>
<td>4861.88</td>
</tr>
<tr>
<td>9</td>
<td>49.875</td>
<td>4876.00</td>
<td>-0.316</td>
<td>4661.88</td>
</tr>
<tr>
<td>10</td>
<td>49.5</td>
<td>4652.25</td>
<td>-0.237</td>
<td>4661.88</td>
</tr>
<tr>
<td>11</td>
<td>49.25</td>
<td>4479.38</td>
<td>-0.395</td>
<td>4513.38</td>
</tr>
<tr>
<td>12</td>
<td>49</td>
<td>4228.75</td>
<td>0.079</td>
<td>4316.38</td>
</tr>
<tr>
<td>13</td>
<td>49.375</td>
<td>3928.88</td>
<td>0.079</td>
<td>3973.38</td>
</tr>
<tr>
<td>14</td>
<td>49.875</td>
<td>3587.38</td>
<td>0.237</td>
<td>3627.75</td>
</tr>
<tr>
<td>15</td>
<td>50.625</td>
<td>3158.50</td>
<td>0.079</td>
<td>3178.88</td>
</tr>
<tr>
<td>16</td>
<td>51.375</td>
<td>2550.00</td>
<td>0.237</td>
<td>2571.38</td>
</tr>
<tr>
<td>17</td>
<td>52.25</td>
<td>1842.25</td>
<td>0.158</td>
<td>1852.13</td>
</tr>
<tr>
<td>18</td>
<td>53</td>
<td>1168.38</td>
<td>0</td>
<td>1172.88</td>
</tr>
<tr>
<td>19</td>
<td>53.625</td>
<td>587.00</td>
<td>0.158</td>
<td>589.88</td>
</tr>
<tr>
<td>20</td>
<td>54.375</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B&amp;L</th>
<th>$V_{\text{expected}}$</th>
<th>50k Monte Carlo Mean</th>
<th>$V_{\text{observed}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implementation</td>
<td>5202.00</td>
<td>5253.82</td>
<td>5168.25</td>
</tr>
</tbody>
</table>

| V_{\text{expected}} | 5255.2 | 5314.9 |

| $V_t/1000$ | 5111.25 | 5111.25 | 5061.25 | 5061.25 | 4961.75 | 4911.88 | 4861.88 | 4661.88 | 4661.88 | 4513.38 | 4316.38 | 3973.38 | 3627.75 | 3178.88 | 2571.38 | 1852.13 | 1172.88 | 589.88 | 0 |

Section 3. Liquidity Models
Table 3.3 compares the output of 50,000 simulations of our recursive implementation of the linear price impact model to a similar simulation done by B&L. Strictly speaking, there should not be any difference between the expected values at $T_0$. Yet, the table clearly demonstrates that there is a slight gap. The observed gap is relatively small (0.35%), and may be attributed to discretization errors. The other differences are more significant, which is as expected as we are considering simulations. However, from the results in table 3.3 it is clear that our implementation produces results of the same magnitude, and thus is most likely correct.

Finally, we want to verify whether the linear percentage price impact model also yields expected results when plugged into our recursive implementation. In the previous section we have replicated the closed-form Monte Carlo simulation by Bertsimas and Lo. Our recursive implementation requires too much computational power to reconstruct the entire set of simulations within an acceptable time frame. Therefore, we have chosen to limit our observations to a single scenario, and make a common sense observation about its validity. Figure 3.3 displays the realized cost savings when the optimal strategy is compared to the naive strategy.

![Histogram of cost savings](image)

**Figure 3.3**: Cost savings with Linear Percentage Price impact, $S = 100,000$, $P_0 = 50$, $\sigma_y^2 = 1 - (0.5^2)$, $\sigma_z^2 = 0.02^2/13$, $\rho = 0.5$, $\gamma = 1$, $\theta = 5 \times 10^{-5}$, $\Delta P = 1/24$, share block size = 5,000, $P$-Discretizations = 160, $X$-Discretizations = 21, $T = 10$, 50,000 Simulations.
From figure 3.3 we can see that the optimal strategy outperforms the naive strategy on average, but does produce extra costs in several cases. The mean cost savings over 50,000 simulations are 11,743.55, or 11.74 ct/share. This is in line with the observations by B&L: their range varies from 0.004 ct/share to over 40 ct/share. At this point we have enough indications that our general implementation works as expected, and thus that we can use it in other applications as well. One relation that has proven crucial in proper execution is the relation between the discretization steps and the volatilities of white noise parameters. It should be no surprise that small volatilities require small discretization steps for the price or state to move away from its original point. Relatively large discretization steps can easily introduce rounding errors which can lead to wrong strategies.

### 3.2.5 Solution procedures for Bertsimas and Lo

In this section we will briefly discuss different solution methods to Bertsimas and Lo’s model. In the previous section we have formulated closed-form analytic solutions, as well as the recursive discretization approach. For larger problems without closed-form solutions the typical discretization approach should not be preferred, as we will see below. The following three methods can be used:

- Closed-form, analytic solutions.
- Recursive, numerical solutions.
  - Discretization.
  - Approximate dynamic programming.

**Closed-form solutions** are rare in the above framework, and in HJB equations in general. Finding laws of motion with closed-form solutions is not a trivial task, and not the goal of this thesis. However, there are other known laws of motion that have closed-form solutions. If we can use one of said laws to model interest-rate products it may be possible to derive a model with a closed-form solution. However, this is far less likely to succeed than one of the numerical approaches.

**Discretization** is a common approach in recursive problem solving, and also does not require further explanation. For this model it is sufficient to take note of the discretization parameters B&L suggest. Three parameters must be discretized: time, state, and price. Optionally the share block size can be discretized too.
Price is discretized by steps of $1/8$, a range up and down from the price at $T_0$. The step size $1/8$ is used because price quotations are usually made in multiples of $1/8$ EUR. Depending on the length of the time interval the width of the interval needs to be determined, longer time horizons require larger price ranges. In the numerical example a range of 10 EUR is used, which corresponds to 80 discretization steps.

Discretization of the state space is less trivial, as it depends on the volatility used in the law of motion. Also, depending on the value of $\rho$, the state space has somewhere between none and instant mean reversion. In most scenarios a range of $\pm 5$ standard deviations will suffice, but for large values of $\rho$ it may need to be larger. Depending on the other parameters and available computational power the number of steps in that range must be chosen. B&L use 10 discretization steps in their numerical examples.

Discretization of time may be related to the number of available trading moments before the offer must be executed. The model does not incorporate transaction fees, which become nontrivial if transactions become relatively small. To avoid problems, it is wise not to use too many trading moments per day. Bertsimas and Lo use 30 minute intervals during the opening time of the stock exchange, which corresponds to roughly 16 trading moments per day.

Setting the share block size larger than 1 can be used to preserve computational power. If for instance 100,000 units of stock are to be acquired it may be sufficient to determine a trading strategy in terms of 10’s or even 100’s of shares. The total number of iterations is a multiple of the above four discretization parameters, and thus choosing this fourth parameter wisely can significantly reduce the required computational power, at the cost of some accuracy.

**Downsides of the discretization approach.** Discretization is far less computationally efficient than approximate dynamic programming. It has the advantage that it returns the exact values on the discretization points rather than approximations, but doing so in large problems with thousands or millions of points can easily become unfeasible. In our use case there are multiple reasons for the problem to become large.

**Long time horizons.** This parameter is unlikely to become crucial in our case, as we are considering a problem where we want to acquire funds fast. Consequently, it is unlikely that the timeframe we are considering will be longer than a couple of days. Still, in the case of only several days to weeks the problem can easily explode, especially if the other discretization variables are chosen such.
Stock volatility. Volatile stock requires larger price ranges, and thus more price discretizations. We will use interest products as the underlying, which are typically less volatile than stock. Still, to simulate liquidity crises, we may want to construct large shifts in price, consequently requiring the discretized price space to be large.

Multidimensional state variables. So far we have considered only one state variable, discretized by ten steps. The model can be extended to incorporate multiple state variables. Two-factor models are commonly used for interest-rate modeling, thus it is not unlikely that we will be using more than one explanatory state variable in our final model.

Approximate dynamic programming. B&L suggest the use of approximate dynamic programming (ADP) to solve their model as the problem becomes larger. Note that there is a trade-off: ADP is significantly faster, but an approximation. As long as the problem can be solved fast enough with the discretization approach this should be preferred above ADP. ADP works by approximating the value function with a quadratic function, which can in its turn always be minimized in closed-form. The approach is formulated in matrix notation, such that it can be used for multidimensional state variables $X$:

\[
Y_t = (P_{t-1}, X_t, W, t)
\]  
(3.55)

At $t = T$ we can compute $V_T(Y_T)$ without minimization step. Now, $V_T(Y_T)$ is approximated by $\hat{V}_T(Y_T)$ as follows:

\[
\hat{V}_T(Y_T) \equiv Y_T'Q_TY_T + b_T'Y_T
\]  
(3.56)

Where $Q_T$ and $b_T$ minimize:

\[
\int_{Y_T} \left( V_T(Y_T) - \hat{V}_T(Y_T) \right)^2 dY_T
\]  
(3.57)

Now, at time $T - k$ we replace $V_{T-k}(Y_{T-k})$ by $\hat{V}_{T-k}(Y_{T-k})$, such that the recursion becomes (we use the general formulation):

\[
V_{T-k}(Y_{T-k}) = \min_{S_{T-k}} \mathbb{E}_{T-k} \left[ f_{T-k}(P_{T-k-1}, X_{T-k}, S_{T-k}, \epsilon_{T-k})S_{T-k} \right.
\]
\[
+ \hat{V}_{T-k+1}(f_{T-k}(), g_{T-k+1}(), W_{T-k} - S_{T-k}) \]
\]  
(3.58)
The minimization in 3.57 is not intensive because of its quadratic form. Once the value of $V_{T-k}(Y_{T-k})$ has been computed it is again approximated by $\hat{V}_{T-k}(Y_{T-k})$ as follows:

$$\hat{V}_{T-k}(Y_{T-k}) \equiv Y'_{T-k}Q_{T-k}Y_{T-k} + b'_{T-k}Y_{T-k}$$

(3.59)

Where $Q_T$ and $b'_T$ again minimize the least squares problem:

$$\int_{y_{T-k}} \left( V_{T-k}(Y_{T-k}) - \hat{V}(Y_{T-k}) \right)^2 dY_{T-k}$$

(3.60)

The ADP procedure can, contrary to the discretization approach, handle multidimensional state problems quite well. Although ADP is an approximation, it is exact up to the point where value functions are at most quadratic. The functions we have seen thus far are quadratic, hence ADP is arguably a better approach than discretization in such cases. Note however that closed-form solution methods, if available, should always be preferred. A downside of ADP is that the magnitude of the approximation error is unknown and not easily measured as it propagates throughout the iterations.

### 3.3 Multiplicative Error Measurement

The model by Bertsimas and Lo has proven quite valuable over the past decade. It has been used in different follow up studies and in attempts to derive liquidity premiums. However, a more recent study [3] proposes a more direct approach. This study, which follows a multiplicative error measurement (MEM) approach, proves to be quite successful in capturing the liquidity dynamics observed in the market. In this section we will discuss this completely different view on modeling liquidity. Later we will decide which basis lends itself best for a model with interest products as underlying.

In their research Engle et. al. [3] choose to model liquidity directly, relating three explanatory variables: market depth, price volatility and depth volatility. These three variables together represent the current state of liquidity of the market, and are related through the following set of equations:

$$X_t = \mu_t \circ \varepsilon_t, \quad t = 1, 2, \ldots, T$$

(3.61)

$$\varepsilon_t \mid \tilde{S}_{t-1} \sim D(1, \Sigma),$$

(3.62)

$$\mu_t = \omega + \sum_{i=1}^{p} A_i X_{t-i} + \sum_{j=1}^{q} B_j \mu_{t-i}$$

(3.63)

In the above set of equations, $\mu_t$ represents the relations between the explanatory variables, and $X_t$ is a three dimensional process of interest. $\mu_t$, $\varepsilon_t$ and $\omega$ are 3x1
vectors, $\Sigma$, $A_i$ and $B_j$ are 3x3 matrices. The symbol $\odot$ denotes the element-wise vector product. The error terms $\varepsilon_t$ follow a conditional distribution with unit mean and support for non-negativity. The researchers argue that this is the best way to ensure non-negativity while still attributing reasonable probability to very small values. Another, more common approach to non-negativity is to take the logarithm of a distribution. This approach does however not result in appropriate probabilities for near-zero values.

When expanded fully, the system becomes as in equations 3.64 - 3.66. Note that $\beta$ is a diagonal matrix.

$$\mu_t^{(1)} = \omega_1 + \alpha_1 X_{t-1}^{(1)} + \beta_1 \mu_{t-1}^{(1)} + \gamma_2 X_{t-1}^{(2)} + \gamma_3 X_{t-1}^{(3)} + \theta' Z_{t-1}$$

$$\mu_t^{(2)} = \omega_2 + \alpha_2 X_{t-1}^{(2)} + \beta_2 \mu_{t-1}^{(2)} + \gamma_1 X_{t-1}^{(1)} + \gamma_3 X_{t-1}^{(3)} + \theta' Z_{t-1}$$

$$\mu_t^{(3)} = \omega_3 + \alpha_3 X_{t-1}^{(3)} + \beta_3 \mu_{t-1}^{(3)} + \gamma_1 X_{t-1}^{(1)} + \gamma_2 X_{t-1}^{(2)} + \theta' Z_{t-1}$$

$t = 1, 2, \ldots, T$

An additional explanatory variable $Z$ is introduced at this point, which allows the model to be calibrated to a far wider range of crises effects. Through this additional variable flights to safety and results of the news impact curve can be included [3].

The market-depth variable can be observed directly, as available trades are written in the limit order book. Volatility of price and depth are not directly observable, and need to be computed. The following two metrics are proposed by Engle et al.

$$Realized \ Volatility \ of \ Price = RV \ P_t = \sqrt{\sum_{k_t=1}^{K} (\Delta P_{k_t})^2}$$

$$Realized \ Volatility \ of \ Depth = RV \ D_t = \sqrt{\sum_{k_t=1}^{K} (\Delta D_{k_t})^2}$$

In the above two metrics $\Delta P$ and $\Delta D$ are shifts in respectively price and depth of the observed item. These figures are readily available, which makes them suitable candidates for the computation of the required metrics. Engle et. al use $K = 300$ seconds as the length of the recent history. The last 300 seconds provide good recent history estimates for volatility which can be used in the MEM framework.

The remaining $\omega$, $\alpha$, $\beta$ and $\gamma$ variables need to be calibrated on market data. Once the model has been calibrated to a time series, relations between the three explanatory variables can be studied, and further simulations can be done to expand the time series into the future.

One of the strengths of this model is that it is able to capture liquidity dynamics quite well, Engle et al. have calibrated the model on different financial crises of recent
history, as well as more regular situations. Their model results are in line with market observations, and can be explained both quantitatively as qualitatively [3]. There are however downsides specific to our application. Expanding the time series into the future requires simulations of the explanatory variables, market depth, and both volatilities. Doing so implies that we are directly simulating liquidity, and are able to study the resulting relations. The objective we have set out is to simulate price movements based on supply or demand, and study the resulting effects on liquidity. The above framework by Engle et al. does not seem suitable for that task.

3.4 Conclusion

In this section we have discussed two classes of liquidity models, the direct approach by Engle et al. and the indirect approach by Bertsimas and Lo. More models exist, but most of them are either variants on the topics we studied or far less advanced.

The model by Bertsimas and Lo can be used to derive optimal execution strategies in situations where purchasing or selling stock influences the current price of that stock. Our goal to model liquidity based on an interest-rate product may be implemented in this framework, although further studies on the behavior of the price of interest-rate products would be necessary. Also, a modification of the model would be necessary such that it derives minimal number of sales to realize a specific amount of cash or cash equivalents, rather than as much cash by selling a specific amount.

The model by Engle et al. seems less suitable for our goal. Although it is able to capture most of the liquidity dynamics observed in the market, its forecasting ability is limited. Forecasting through this model is done by directly simulating time series for market depth, volatility and price volatility. Doing so would not yield the results we are interested in, as we want to study liquidity as a result from market processes, rather than by directly simulating it’s characteristic parameters.

From this point onward we will focus on finding ways to transform the model by Bertsimas and Lo such that it can be used with an underlying interest product, rather than equity. To do so, we will first study different interest models in the next section, and find ways to combine the two in subsequent sections.
Section 4

Interest Rate Models

In this section we will discuss some of the interest-rate models that are widely used in practice. Interest-rate modeling is a crucial component in many valuations across sectors, and as such a lot of research has been done on the topic. As a result, many different models have been developed, one better than the other. However, there is no industry-wide adoption of a single model, because depending on the purpose, different models have better characteristics. We will first discuss the more simple Hull-White one-factor model, before moving on to two-factor models. Finally we briefly touch upon the BGM model, more commonly referred to as the Libor Market Model (LMM).

4.1 Hull-White one-factor model

The one-factor Hull-White model is the first version of a series of models developed by John Hull and Alan White. Although it was developed in 1990, it is still used a lot in the present day. This is primarily because of its relative simple implementation, while it can still be calibrated to match the term structure of interest rates. Good calibration on the term structure is a requirement for correct pricing of interest-rate products such as swaptions. In its most general form the one-factor Hull-White model is defined as follows:

\[ dr(t) = (\theta(t) - \alpha(t)r(t)) \, dt + \sigma dW(t), \quad t > 0 \]  

(4.1)

In the above equation \( r \) represents the interest rate, \( \theta \) some (time dependent) mean-reversion factor, and \( \alpha \) the mean-reversion rate. \( W \) is a standard Wiener process. The above implementation is known as a mean-reverting Ornstein–Uhlenbeck process. Industry practice is to calibrate \( \theta \) on the present-day yield curve, while \( \alpha \) and \( \sigma \) are estimated from historical data.
The following three special cases of general Hull-White one-factor models exist:

- $\theta$ and $\alpha$ constants, the Vasicek model.
- $\theta$ $t$-dependent, $\alpha$ constant, the standard Hull-White model.
- $\theta$ and $\alpha$ $t$-dependent, the extended Vasicek model.

The Hull-White model is one of the most widely used single-factor interest-rate models. Due to several interesting properties, next to its relatively simple implementation it can be very useful. The properties we are referring to here are the following:

- Gaussian distribution, and thus high analytic tractability allows for closed-form expressions for many option valuations.
- Market data suggests that interest rates do indeed revert to a mean level, which is captured by the model.

The models above do however all share one negative property: the interest rate can decrease below zero, something that is hardly seen in practice. Other versions of one-factor models resolve said issue by taking the exponent of the process, but this implies concessions on analytic tractability.

One-factor models have another issue that is problematic for our purposes: because of the single random shock $W$, the changes along the yield curve are perfectly correlated. This means that once the model has been calibrated to the present-day yield curve, the shape of the yield curve remains fixed. In other words, one-factor models can only account for parallel shifts in the yield curve, while in practice other shifts occur as well. We are interested in simulating a crisis where short-term funds are no longer available at normal prices, while long-term interest rates will see limited effects. This implies a change in the yield curve which is far from parallel.

### 4.2 Hull-White two-factor model

Two-factor models have been developed to overcome some of the shortcomings of one-factor models. Two-factor models typically are of the following form:

$$dx(t) = k_x(\theta_x - x(t))dt + \sigma_x dW_1(t)$$  \hspace{1cm} (4.2)

$$dy(t) = k_y(\theta_y - y(t))dt + \sigma_y dW_2(t)$$  \hspace{1cm} (4.3)

$$dW_1 dW_2 = \rho dt$$  \hspace{1cm} (4.4)

$$r_t = x_t + y_t, \quad t > 0$$  \hspace{1cm} (4.5)
Since there are two ($\rho$-correlated) Wiener processes involved in the computation of $r$, this model does account for changes in the shape of the yield curve. Again, the above implementation can result in values below zero which can be solved by taking the exponent of the process, or making other modifications (i.e. explicitly imposing a non-negativity condition). In the above form the model keeps most of its analytic tractability however, which is often preferred above the exclusion of negative rates.

The above model can clearly be extended to multi-factor models, simply by adding a $dz$ term depending on a third Wiener process possibly correlated to the other Wiener processes. Such multi-factor models are being used in the industry, but two-factor models have a far higher penetration rate. This is mostly because there is no clear economic reason for adding another process and thus extra variables, and as such chances are that the variables are being over-fitted to current market data.

4.3 Libor Market Models

A third class of interest-rate models that have a high penetration rate are the Libor Market Models (LMM). Libor market models are far more complex in nature than both one and two-factor models, but have the advantage that they are able to completely rule out arbitrage possibilities in the interest-rate simulations. This property can be of crucial interest in valuation issues, where the deviations in the final price have direct, significant impact. This additional property does however come at a price, LMM-models are far more complex to implement, and computationally far more intensive than the models we have seen earlier.

Where in other models either the short rate or a single forward rate is modeled, LMM-models model the whole set of forward rates under a common measure. These all follow lognormal processes in the basic LMM-model, and have the advantage that they are all directly observable in the market at time zero. Payoffs of interest products are all translated into payoffs of forward rates, and can thus be priced accordingly. The model can be calibrated to widely available market data, and further simulations do not introduce arbitrage because they run under the same measure. Therefore, LMM-simulations are the best choice for pricing complex products.
The model dynamics are as follows:

\[
dL_j(t) = \sigma_j(t) L_j(t) dW_Q^{\tau_j}(t), \quad t > 0
\]  

(4.6)

\[
W_Q^{\tau_j} = \begin{cases} 
    dW_Q^{\tau_p} - \sum_{k=j+1}^{p} \frac{\delta L_k(t)}{1+\delta L_k(t)} \sigma_k(t) dt & j < p \\
    dW_Q^{\tau_p} & j = p \\
    dW_Q^{\tau_p} + \sum_{k=p}^{j+1} \frac{\delta L_k(t)}{1+\delta L_k(t)} \sigma_k(t) dt & j > p
\end{cases}
\]  

(4.7)

Looking at the benefits of LMM-models, one would think it is best to use one for every interest rate related issue. In reality we do however still see a high adoption of both one and two-factor models, mostly due to the complexity and computational requirements of LMM-models. For our purpose, the one-factor Hull-White model has too many shortcomings to be seriously considered. The two-factor model family does however offer the additional features we require (wider variation in yield-curve shapes). The additional benefits of LMM-models are most interesting in pricing complex interest products, something we will not be doing. Because of the relative ease of implementation we have chosen to use the one-factor models as a starting point. These models are expected to provide limited results, and we will therefore quickly move on to the two-factor Hull-White family of models. At a later time we may choose to extend our model to the LMM-family, however at this time it seems unfeasible.
Section 5

Model Implementations

In this section we will discuss different approaches through which we can combine interest-rate models with the Bertsimas and Lo liquidity model (BL-model). We will try several direct approaches, where we replace parts of the BL-model it captures the dynamics of the interest rate. Forms of the one and two-factor Hull White model will be considered, the LMM-model is deemed too complex for this purpose. Alternatively we can choose for an indirect approach. Rather than including the interest rate itself in the BL model, we use an interest-rate product. Then, based on resulting fluctuations in the price of that product we reconstruct the yield curve, and derive the implied interest rate. Summarizing we will implement models of the following forms:

- Models where the HW1f-model is introduced at different points in the BL-model. Directly simulate changes in the interest rate based on demand for the interest rate.

- Models where the HW2f-model is merged with the BL-model in different ways. Directly simulate changes in the interest rate based on demand for the interest rate.

- An indirect approach where the BL-model is used to model price effects of liquidity on an interest-rate product. The yield curve is reconstructed to observe the impact.

Recall that Hull-White models are used to model the instantaneous forward rate at time $t$. Now, as we are taking $t$ small because we are considering a short period in which the funds must be attracted, these models will only produce interest-rate shifts for a short period. It is therefore unlikely that these models will be a good approach for the funding problem with longer maturity. On the other hand, direct simulations on
products is expected to be more valuable in the longer term, because most frequently traded bonds have a maturity of at least one year as we will see later.

5.1 One-factor Hull-White model implementations

This approach to modeling liquidity requires few modifications to the original BL model. We can model movements in the interest rate through the Hull White model, and substitute that directly in the state variable of the BL-model. In the original BL-model the relation in the state variable is as in equation 5.1.

\[ X_t = \rho X_{t-1} + \varepsilon_t, \quad t > 0 \]  

(5.1)

We can write the relation of the HW1f-model instead. Now we combine the functions for \( X_t \) and \( P_t \) into a single function. The first price function for a combined model under linear price impact assumptions then becomes as follows:

\[ P_t = P_{t-1} + \left( \theta^{(1)}(t) - \alpha P_{t-1} \right) dt + \sigma dW(t) + \theta^{(2)} S_t + \varepsilon_t, \quad t > 0 \]  

(5.2)

Note that the constant \( \gamma \) vanishes, as we now directly simulate interest-rate movements and thus do not require further multiplication. In equation 5.2 effects of the demand \( S_T \) are "permanent", although there is the mean-reverting term that counters changes in due time. The impacts are otherwise unrelated to the underlying interest-rate process, a fact which is arguably wrong. In reality interest-rate shifts in distressed situations can trigger flights to liquidity. Such behavior can be included in the model by making the mean-reverting term \( \theta^{(1)}(t) \) dependent of the demand. Recall that the \( \theta^{(1)}(t) \) term is chosen such that the interest process fits the present-day yield curve. We add an additional term to the calibrated \( \theta^{(1)}(t) \) term, which increases the mean reversion in times of increasing demand. Note that the proposition below is an extension of the 1 factor model, and thus in fact more close to the two-factor model:

\[ P_t = P_{t-1} + \left( \theta^{(1)}(t) - \alpha P_{t-1} \right) dt + \sigma dW(t) + \theta^{(2)} S_t + \varepsilon_t, \quad t > 0 \]  

(5.3)

\[ d\theta^{(1)}(t) = d\theta^{(1)}(t) + \theta^{(2)} S_t \]  

(5.4)

Now, before continuing towards two-factor versions of the Hull White model, let us first consider extensions in the form of the linear percentage temporary price impact model. B&L concluded that their first two models were still very limited, and the above propositions inherit those limitations as well. Recall that the linear percentage
temporary price impact has the following characteristics:

\[ P_t = \tilde{P}_t + \Delta_t, \quad t > 0 \]  
\[ \tilde{P}_t = \tilde{P}_{t-1} \exp(Z_t), \quad Z \sim N(\mu_z, \sigma_z) \]  
\[ \Delta_t = (\theta S_t + \gamma X_t) \tilde{P}_t \]  
\[ X_t = \rho X_{t-1} + \eta_t \]  

Now, including the HW one-factor interest function can be done at multiple levels. Implementing it at the \( \tilde{P} \)-level would mean that, contrary to what we have seen above, purchases would not influence the interest rate through the drift term, but only through the temporary \( \Delta_t \) term. Impacts of purchases will therefore be smaller, all other parameters remaining the same. Such an implementation would follow the relation below, again freeing the state variable \( X \) as we now include the interest process in the permanent price process \( \tilde{P} \):

\[ P_t = \tilde{P}_t + \Delta_t, \quad t > 0 \]  
\[ \tilde{P}_t = \tilde{P}_{t-1} + \left( \theta^{(1)}(t) - \alpha \tilde{P}_{t-1} \right) dt + \sigma dW(t) \]  
\[ \Delta_t = \left( \theta^{(2)} S_t + \gamma X_t \right) \tilde{P}_t \]  
\[ X_t = \rho X_{t-1} + \eta_t \]  

Another, possibly more realistic approach would be to let the \( S_t \)-term have an effect on the \( \tilde{P} \)-term as well. This can be done directly, or through a manipulation of the mean-reversion term such as we have seen above. The following two possible implementations come to mind:

\[ P_t = \tilde{P}_t + \Delta_t, \quad t > 0 \]  
\[ \tilde{P}_t = \tilde{P}_{t-1} + \left( \theta^{(1)}(t) - \alpha \tilde{P}_{t-1} \right) dt + \sigma dW(t) \]  
\[ \Delta_t = \left( \theta^{(2)} S_t + \gamma X_t \right) \tilde{P}_t \]  
\[ d\theta^{(1)}(t) = \bar{d}\theta^{(1)}(t) + \theta^{(2)} S_t \]  
\[ X_t = \rho X_{t-1} + \eta_t \]  

\[ \text{or} \quad P_t = \tilde{P}_t + \Delta_t, \quad t > 0 \]  
\[ \tilde{P}_t = \tilde{P}_{t-1} + \left( \theta^{(1)}(t) - \alpha \tilde{P}_{t-1} \right) dt + \theta^{(2)} S_t + \sigma dW(t) \]  
\[ \Delta_t = \gamma X_t \tilde{P}_t \]  
\[ X_t = \rho X_{t-1} + \eta_t \]
We now have several different ideas through which the Hull-White one-factor model can be merged into the model by Bertsimas and Lo. Experiments will be conducted to find the model that most closely resembles reality. Since Bertsimas and Lo have already concluded that their first two implementations are not fit to model reality we will restrict our experiments to models of the temporary price impact model. In section 7.1 we will review the results of our experiments, but for now we have the following hypotheses:

- Models without drift influence do not properly simulate the aftershock in interest-rate shifts; and
- Models with permanent drift influence have too high aftershock effects.

### 5.2 Two-factor Hull-White model implementations

In the Hull-White two-factor model the most common clarification for the second factor is the different behavior of long-term versus short-term interest rates. Long-term interest is far less volatile, and is often assumed to be constant for long periods of time. Two-factor models are able to capture the changes in the shape of the yield curve that present themselves when the short-term interest rates fluctuate while longer term rates remain constant.

In our use case the most natural modeling assumption would be to have demand for interest rate affect one factor of the model, leaving the other uninfluenced. Recall the dynamics of the two-factor model:

\[
\begin{align*}
    dx_t &= k_x (\theta - x_t) dt + \sigma_x dW_{1,t}, \quad t > 0 \\
    dy_t &= k_y (\theta - y_t) dt + \sigma_y dW_{2,t} \\
    dW_1 dW_2 &= \rho dt \\
    r_t &= x_t + y_t
\end{align*}
\]
Now, we have two options through which purchases can influence $x_t$, leaving $y_t$ uninfluenced:

\[
\begin{align*}
\dot{x}_t &= k_x(\theta_x - x_t)dt + \sigma_x dW_{1,t}, \quad t > 0 \\
\dot{y}_t &= k_y(\theta_y - y_t)dt + \sigma_y dW_{2,t} \\
\dot{W}_1 \dot{W}_2 &= \rho \, dt \\
r_t &= x_t + y_t
\end{align*}
\]  

or \[
\begin{align*}
\dot{x}_t &= k_x(\theta_x - x_t)dt + \sigma_x dW_{1,t}, \quad t > 0 \\
\dot{\theta}_x &= \dot{\theta}_x + \theta S_t \\
\dot{y}(t) &= k_y(\theta_y - y_t)dt + \sigma_y dW_{2,t} \\
\dot{W}_1 \dot{W}_2 &= \rho \, dt \\
r_t &= x_t + y_t
\end{align*}
\]

Where again in the first version we see a limited impact of demand, while in the second version demand permanently increases the mean-reversion level. The hypotheses stated above apply to these modeling implementations as well. Limiting the impact of purchases to the current time unit (as in the first model) may fail to capture the aftershock of interest-rate shifts. On the other hand adding it to the $d\theta$ term permanently increases the mean-reversion level, meaning that the interest rate would never recover from purchases made. Possible solutions that we will consider are combinations of the above two models, and a slow mean-reverting element in the mean-reverting $d\theta$-term.

### 5.3 Hull-White calibration

In our model suggestions we have added additional variables to the one and two-factor Hull-White models. This also means that an additional degree of freedom has been introduced, one that makes the already difficult task of model calibration more complex. We do however have one advantage that reduces this problem significantly: The additional variable is only in effect when parts of the order are being executed. This means that we can first calibrate the models as we would under standard Hull-White model calibration, and subsequently calibrate the $\theta^{(2)}$-term.

Note that we do not need to calibrate the models to specific market conditions to be able to compare the models. Any reasonable choice of variables (one that could present in reality) is sufficient to qualitatively conclude on the performance of our models. Ideally we would want a quantitative analysis, but this would require isolated liquidity
premium data from similar situations, which is not available. We could attempt to extract this information from Bloomberg, but because of the correlation with other problems it is unlikely that we will find a scenario to which we can calibrate our models without introducing significant errors. It is for this reason that we have chosen for qualitative comparison.

One-factor Hull-White calibration can be achieved by following the steps below. First recall the dynamics of the model:

\[ dr_t = (\theta_t - \alpha_t r_t) dt + \sigma dW_t, \quad t > 0 \]  

In equation 5.35 three variables have to be calibrated. The first step is the calibration of the \( \alpha \)-term. This is done by setting:

\[ \alpha_t = f(0, t) + \frac{1}{E(t)} \int_0^t E(u) \sigma^2(u) B(u, t) du \]  

With \( f(0, t) \) the instantaneous forward rate at \( t = 0 \), \( E(t) \) the integration of \( \alpha_t \), and \( B(t, T) = E(t) \int_t^T \frac{du}{E(u)} \). Next, the \( \sigma \)-term is extracted from either captions or swaptions (interest products) that are actively traded in the current market. This step requires no further mathematical explanation. Finally, the \( \theta \)-term is calibrated such that the model replicates the term structure at \( t = 0 \):

\[ \theta_t = \frac{\delta}{\delta t} f(0, t) + \alpha_t f(0, t) + \frac{1}{2} \left( \frac{\delta^2}{\delta t^2} V(0, t) + \alpha_t \frac{\delta}{\delta t} V(0, t) \right) \]  

Where \( V(t, T) = \int_t^T \sigma^2(u, T) du \) and \( \sigma(u, T) = \sigma(u) B(u, T) \).

The above calibration method is explained more thoroughly in [4]. For the purpose of this research it suffices to understand that the model can be calibrated to current market conditions by following these steps, but we will not actually implement this procedure for comparison.

Two-factor Hull-White calibration. Recall the dynamics of the two-factor Hull-White model:

\[ dx_t = k_x (\theta_x - x_t) dt + \sigma_x dW_{1,t}, \quad t > 0 \]  
\[ dy_t = k_y (\theta_y - y_t) dt + \sigma_y dW_{2,t} \]  
\[ dW_{1}dW_{2} = \rho dt \]  
\[ r_t = x_t + y_t \]
As is clear from the above equations, far more parameters need to be calibrated in two-factor models. Different approaches to this calibration exist, none of which are trivial. Since we will not be calibrating our models on market data in this research we believe an in dept review of the calibration of this model is therefore redundant. The interested reader should refer to [5] for a full explanation of the calibration of the two-factor additive Gaussian model. This model is in fact equivalent to the two-factor Hull-White model.

**Calibration of the \( \theta^{(2)} \)-term.** Once the \( \alpha, \sigma \) and \( \theta \) terms are calibrated we can heuristically calibrate the remaining \( \theta^{(2)} \)-term. Note that this term is the liquidity dependency parameter, which is situation specific. We can therefore not calibrate it on historic data, but at best choose a smart starting point based on previous executions. We can however beforehand compute the implied full impact of a given theta, which is simply \( \frac{\theta^{(2)} \delta}{\theta^{(0)}} \). The steps to follow to quickly derive this parameter by following the recursive learning procedure below:

- Based on the market conditions choose three \( \theta^{(2)} \)-values with implied full impact spread around the expected liquidity premium that would be paid if all was executed at once.
- Run the model with all parameters. This results in several strategies, each optimal if its \( \theta \) is correct.
- Execute the first step in the (conservative) strategy. This immediately yields the exact price bump, plus or minus some error and spread.
- Dismiss the previous strategies, and rerun the model with \( \theta^{(2)} \)-term as by the observed premium.
- As more orders are executed more information on the \( \theta^{(2)} \)-term becomes available. If it still deviates from the model implied premiums, re-apply the steps.

The following notes must be made:

- We have taken three initial guesses for the \( \theta^{(2)} \)-term. Depending on the available computational power, insight in the situation and severity of the situation this number can be varied. Increasing the parameter is only useful for deciding the size of the first executed order, after this point a single new strategy suffices.
- It is wise to choose the initial spread of the \( \theta^{(2)} \)-term large if no estimation on the full liquidity impact can be made. Under estimating the term can lead to unexpected losses which should be averted at all times.
• Choosing a conservative strategy for the first execution results in less extreme losses if the term is far off. The rate of convergence is unchanged, hence this choice.

• We assume the $\theta^{(2)}$-term to be constant in the model. More advanced models may call for time dependency, which would further complicate the calibration.

### 5.4 Yield-curve construction

The methods that we have discussed in the previous two subsections share the same flaw: they model the demand for the interest rate. However, direct investments in the interest do not exist in real-life markets. Trading is done through interest-rate products, which vary from complex financial constructions to simple loans. Another approach at modeling liquidity is to simulate the price that must be paid for a common interest-rate product. Then, based on the changes in the value of that product due to supply constraints, the yield curve can be re-constructed and new interest-rate levels can be derived. This different approach has the advantage that there is no need to manipulate the model by B&L. For some frequently traded interest-rate products there is no reason to assume that the liquidity premium that must be paid behaves differently from equity. We are interested in the interest-rate products which have very short maturity, as those are the products that banks use to re-roll their overnight loans. Shortages in those products cannot easily be replaced by other financial instruments other than fire-sales or government bailouts, precisely those events which typically occur in crises.

#### 5.4.1 Interest product candidates

There are multiple candidate products to select from, and modeling more than one product simultaneously does not easily fit the model by B&L. We will discuss the most likely candidate products shortly, before selecting one to be used in our model.

**Repurchase agreements**, commonly referred to as repo agreements, are the sale of a set of securities with a fixed repurchase rate and date. The repurchase price is higher, the difference represents the interest (repo rate) the lending party receives for temporarily buying the securities. Banks rely on repos to re-roll their overnight loans, such that they are able to meet short-term financial obligations. Repo rates are usually around 200 bp below regular interest rates, because of the limited risk involved due to the repurchase agreement. However, in distressed situations risk increases significantly,
and far less counter parties willing to participate in a repo agreement can be found. This triggers an increase in the repo rates, and immediate liquidity problems as these instruments are used to meet next-day requirements.

Repos are split into two categories: overnight repos and term repos. Overnight repos are commonly used to fill one day liquidity mismatches. Term repos usually have terms of several months maximum. Overnight repos are the product of interest to our use case, if that market dries up or becomes increasingly expensive it poses immediate liquidity problems. We would however have to consider the whole overnight repo market as a single product, which is a grave generalization: repo agreements have multiple elements to them other than term and rate. The amount of collateral that must be posted by the borrowing partly can be increased to reduce risk, and so called margin agreements can also be included to reduce the lenders exposure. We will have to ignore these two components if we wish to simulate purchase effects on the price of repo rates. Repo rates are not quoted, but can be derived from non-arbitrage rules. The following portfolio’s payoff should be equal to the repo rate and is therefore called the implied repo rate:

- Sell a futures contract of stock X;
- Buy a bond of stock X with the obtained money;
- At expiry, use the stock to pay off the future debt; and
- Implied repo rate = \( \text{premium} \times \frac{360}{\text{timeheld}} \)

**Commercial Paper** (CP) are, opposed to repo agreements, fully unsecured loans. Consequently interest rates are higher, and only within reasonable levels for companies with high credit ratings. Interest on commercial paper increases with maturity, which is usually no more than 270 days. Although CP has no collateral to limit the lenders exposure, highly rated companies can often get rates slightly below bank loans. Such financial incentives make CP an interesting instrument for borrowers, and lenders do not necessarily require collateral from highly rated companies if terms are short. From a modeling perspective an advantage of CP over repo rates is that less variables are involved, and thus less generalization is required. However, if we wish to model CP with a specific term as a single instrument we still need to generalize over all credibility groups. Doing so is clearly wrong when we review the CP rates: the 1 day CP rate can be as low as 0.03%, but also as high as 0.28%. Instead, we may use CP for a specific credibility group such as AA or AAA, limiting our model results to that group. Another option would be to use our model at the country level, which requires the least
generalization. If this approach is followed we may use actual, historic CP quotations as a starting point. A downside is that this information may not be readily available as not all companies are actively trading CP.

A second advantage from a modeling perspective is that CP figures are quoted on the credibility group level, which means that less effort is needed to find the required data compared to repo rates.

High-frequency short-term bonds. A final option is to use frequently traded bonds with short maturities in our model. Although this variant would use less country specific information, the bond market is arguably the best representation of present day interest rate. Information for bonds is readily available, and can be traded between any two parties. This third method is the most straightforward implementation, but may in some situations not provide the best representation of the interest rate that the party would have to pay in reality.

Conclusion. From the above three methods the high-frequency short-term bonds are clearly the most easy to implement, and applicable to any party for which the model is applied. Repo’s are unlikely to be preferred to Commercial Paper, except in situations where the country is not able to write CP at an attractive price. In those situations it is however unlikely that the country’s financial requirements are able to disrupt the funding market. For some specific scenarios CP will work better than short-term bonds evaluations, however we prefer to set-up our model in the general sense and note that for further tailoring country specific interest-rate quotations can be used, if available.

5.4.2 Yield-curve construction methods

Yield curves display the continuously compounding interest rate for given maturity levels. Theoretically, yield curves can easily be derived from zero coupon bonds, as long as we assume the no arbitrage assumption holds. If we then write $Z(0, t)$ for the value of a zero coupon bond with maturity $t$, we can retrieve the continuously compounding interest rate for period $(0, t)$ as follows:

$$r(t) = -\frac{1}{t} \ln Z(0, t), \quad t > 0$$ (5.42)

In a market where zero coupons for all maturities of interest are actively traded the above formula may suffice. However, in reality there are two complications. First, zero coupons are not actively traded and thus price quotations are not always accurate or recent. Second, for many purposes a continuous yield curve is much more appropriate than a
stepwise function. Clearly not all points on a continuous graph have a corresponding quoted zero coupon, hence the need for interpolation methods.

Patrick Hagan [6] describes and compares different yield-curve construction methods. Moreover, he has created a framework through which different methods can be compared, and thus selected based on the application’s requirements. We will not go into all different approaches, for that the reader should refer to [6]. Instead we will briefly review his framework and the different categories he identifies within the group of construction methods. Methodologies are scored on the following five criteria:

- **Forwards positive.** A good method should not be able to return negative forward rates in any time frame to avoid arbitrage opportunities;

- **Forward smoothness.** A continuous curve is preferred, as interest-rate sensitive products can show large price fluctuations otherwise;

- **Method locality,** changes in values should have most of their impact be local, such that substantial changes along the curve due to a single change are minimized;

- **Forwards stability.** While continuity is one requirement, stability is also highly desirable for most purposes; and

- **Bump hedges locality.** The delta-risk in a hedged portfolio should be assigned to the instruments that have maturities close to the tenors of the hedge instruments.

Hagan describes linear methods, as well as splines and monotone convex methods, which we will not discuss further. His results are displayed in table 5.1.

<table>
<thead>
<tr>
<th>Yield-curve type</th>
<th>Forwards positive</th>
<th>Forwards smoothness</th>
<th>Method local</th>
<th>Forwards stable</th>
<th>Bump hedges local?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear on discount</td>
<td>no</td>
<td>not continuous</td>
<td>excellent</td>
<td>excellent</td>
<td>very good</td>
</tr>
<tr>
<td>Linear on rates</td>
<td>no</td>
<td>not continuous</td>
<td>excellent</td>
<td>excellent</td>
<td>very good</td>
</tr>
<tr>
<td>Raw (linear on log of discount)</td>
<td>yes</td>
<td>not continuous</td>
<td>excellent</td>
<td>excellent</td>
<td>very good</td>
</tr>
<tr>
<td>Linear on the log of rates</td>
<td>no</td>
<td>not continuous</td>
<td>excellent</td>
<td>excellent</td>
<td>very good</td>
</tr>
<tr>
<td>Piecewise linear forward</td>
<td>no</td>
<td>continuous</td>
<td>poor</td>
<td>very poor</td>
<td>very poor</td>
</tr>
<tr>
<td>Quadratic</td>
<td>no</td>
<td>continuous</td>
<td>poor</td>
<td>very poor</td>
<td>very poor</td>
</tr>
<tr>
<td>Natural cubic</td>
<td>no</td>
<td>smooth</td>
<td>poor</td>
<td>good</td>
<td>poor</td>
</tr>
<tr>
<td>Hermite/Bessel</td>
<td>no</td>
<td>smooth</td>
<td>very good</td>
<td>good</td>
<td>poor</td>
</tr>
<tr>
<td>Financial</td>
<td>no</td>
<td>smooth</td>
<td>poor</td>
<td>good</td>
<td>poor</td>
</tr>
<tr>
<td>Quadratic natural</td>
<td>no</td>
<td>smooth</td>
<td>poor</td>
<td>good</td>
<td>poor</td>
</tr>
<tr>
<td>Hermite/Bessel on rt function</td>
<td>no</td>
<td>smooth</td>
<td>very good</td>
<td>good</td>
<td>poor</td>
</tr>
<tr>
<td>Monotone piecewise cubic</td>
<td>no</td>
<td>continuous</td>
<td>very good</td>
<td>good</td>
<td>good</td>
</tr>
<tr>
<td>Quartic</td>
<td>no</td>
<td>smooth</td>
<td>poor</td>
<td>very poor</td>
<td>very poor</td>
</tr>
<tr>
<td>Monotone convex (unameliorated)</td>
<td>yes</td>
<td>continuous</td>
<td>very good</td>
<td>good</td>
<td>good</td>
</tr>
<tr>
<td>Monotone convex (ameliorated)</td>
<td>yes</td>
<td>continuous</td>
<td>good</td>
<td>good</td>
<td>good</td>
</tr>
<tr>
<td>Minimal</td>
<td>no</td>
<td>continuous</td>
<td>poor</td>
<td>good</td>
<td>very poor</td>
</tr>
</tbody>
</table>
For most applications arbitrage free methods are strongly preferred over other methods. In table 5.1 only three candidate methods have this property, of which the unameliorated monotone convex method scores best. Ideally we would want to implement the monotone convex algorithm for the construction of the yield curve. This method is however not as easily implemented as cubic spline interpolation, which offers a good starting point. Since cubic spline interpolation methods are readily available we will use those for a first review of our proposed models. We may implement the monotone convex interpolation later, should the Bertsimas and Lo approach prove far less realistic than the indirect yield curve approach.
Section 6

Yield Curves in Practice

In previous sections we have described liquidity and interest models, and proposed different approaches to combining the two. Before we are able to say anything about the quality of the models we need to study empirical behavior of the yield curve. We can only comment on the yield curves our models produce once we have a clear understanding of how yield curves have behaved historically.

We will discuss yield curves of different instruments during the US recession of 1991. We will consider the curves before the crisis, during the crisis, and finally when the crisis has subsided. The generally accepted theory about the shape of yield curves is that it depends on two components:

- Investors expectation of future interest rates.
- Risk premiums that investors expect for holding long-term bonds.

In a healthy economy, investors typically expect the interest rate to rise in the future, therefore the shape of the yield curve becomes asymptotically upwards sloping. Figure 6.1 displays a typical shape in a healthy economy.

![Figure 6.1: Example yield curve in normal economy](image.png)
Now, let us consider yield curves of US Treasury bonds around the US recession of 1991 to get an idea of how economic slowdown and recovery are reflected in the yield curve. We first consider the period prior to the recession. The Salomon yield book has data some time before the recession, 31/12/1989.

The figure on the left in 6.2 illustrates how yield curves can flatten out when investors expect an economic slowdown. They no longer expect the interest rates to rise in the near future, and as a consequence the return on long-term bonds decreases. As the recession nears, investors expect long-term interest rates to drop further. Around 12 to 18 months before the recession the yield curve can become inverted, signaling an upcoming recession. Figure 6.2(right) displays the inverted yield curve around a year before the recession of 1991.

Due to the same mechanism yield curves steepen when investors expect economic recovery. Interest rates are expected to rise more sharply than in a regular economy, because recovery to the original levels is imminent. Figure 6.3 displays the yield curve on US Treasuries shortly after the recession, 30/4/1992.

A fourth, less common shape is the so-called humped yield curve. Humped yield curves are those where the yield on midterm bonds is higher than returns on both low and long-term bonds. In recent years, humped curves have become more common however, due to the high demand for 30-year bonds. High demand for those types of bonds have pressed their prices, driving the yield below the 20-year bonds. An example of a humped yield curve is displayed in figure 6.4.
Ideally, the output of our models should produce yield curves that could be observed in reality. As illustrated above this means that we expect our models to produce normal yield curves in situations where the impact of purchases is minimal to none, flattening yield curves when there is some impact, and finally inverted yield curves in situations where the impact is high. However, we are now referring to the effects that persist after the execution of the order, once the yield curve has stabilized. We also expect a temporary spike at the time of execution, which results in yield curves that do not persist long after the execution of the order. These temporary effects are due to the liquidity premium that must be paid, and is in fact the objective of (weighted) minimization in the optimal execution strategy. We expect that yield curves that are constructed precisely at those execution moments are more 'extreme' than we expect to observe in reality, because the temporary liquidity impact is included in the curve. We use the term 'implied yield curve' for curves that would present if the present interest rates would persist, but likely won’t because the situation has not yet stabilized. Stabilization automatically occurs after financial disruption as the market seeks an equilibrium where no arbitrage opportunities exist. Temporary arbitrage opportunities often present after heavy market disruptions, in this case the market could have yield curve arbitrage opportunities.

We will see that the Hull White model variants realize yield-curve shifts that consist mostly of the temporary spike, which has limited effect on its persisting shape. Direct simulations on interest products on the other hand will realize the shifts we expect to observe in longer term funding problems. This is in line with the expectation that different models serve better in specific situations.

Unfortunately, there is no way to determine the exact error, because there is no quantitative data available on how our models should react under different conditions. This is mostly so because in reality these events do not occur isolated, but rather as a component of a more complex scenario. Proper quantitative analysis is therefore impossible. However, we can compare the results between models qualitatively and draw conclusions on the resulting shapes of the yield curves and their plausibility.
Section 7

Model Comparison

In this section we will review the results of the models from section 5. As we noted earlier, the Hull-White family of models is better suited for the short-term cases, while the direct simulations on the interest products are better suited for longer term scenarios. This poses a problem, as comparing results now is non-trivial. For longer term problems we can reconstruct the implied yield curve, and comment on its likelihood. However, for shorter term problems the yield-curve shifts will be minimal, as we only modify data very early in the curve. Interpolation will not change much in such cases, as we will see in sections 7.1 and 7.2.

We have decided to evaluate the Hull-White models independently of the direct simulations, using a numerical approach in which we will comment on the paths the instantaneous forward rate followed during the execution period. In the concluding section of this thesis we will present our views on which models can be used in practice, and in what situations.

Before going into the reviews we present the data set that has been used throughout this section. It consists of actively traded government bonds in three countries, Germany, Austria and France. The data can be found online in the termstrc package which is available as a plug-in for R. A brief summary of the data is given in table 7.1. This data-set will be used in the Hull-White model variants, as well as for direct simulations on interest products.

We have decided to use Nelson-Siegel (NS) interpolation, a monotone convex interpolation method, rather than cubic spline interpolation. The additional value of this choice should be clear from section 5.4.2, however a more illustrated explanation of this choice can be found in Appendix B. The results of NS-interpolation on the data-set provides
Table 7.1: Summary of data sets used in the yield-curve construction method.

<table>
<thead>
<tr>
<th>Country</th>
<th>Germany</th>
<th>Austria</th>
<th>France</th>
</tr>
</thead>
<tbody>
<tr>
<td>Date:</td>
<td>2008-01-30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of bonds</td>
<td>52</td>
<td>16</td>
<td>45</td>
</tr>
<tr>
<td>Expiry Min</td>
<td>15-2-2008</td>
<td>15-7-2009</td>
<td>12-3-2008</td>
</tr>
<tr>
<td>Max</td>
<td>4-7-2039</td>
<td>15-3-2037</td>
<td>25-4-2055</td>
</tr>
<tr>
<td>Issue date Min</td>
<td>20-6-1986</td>
<td>10-7-1997</td>
<td>26-1-1987</td>
</tr>
<tr>
<td>Max</td>
<td>21-9-2007</td>
<td>8-1-2008</td>
<td>8-6-2007</td>
</tr>
<tr>
<td>Min</td>
<td>0.025</td>
<td>0.35</td>
<td>0.25</td>
</tr>
<tr>
<td>Coupon Mean</td>
<td>0.043</td>
<td>0.044</td>
<td>0.04</td>
</tr>
<tr>
<td>Max</td>
<td>0.065</td>
<td>0.063</td>
<td>0.085</td>
</tr>
<tr>
<td>Min</td>
<td>91.56</td>
<td>91.11</td>
<td>88.97</td>
</tr>
<tr>
<td>Price Mean</td>
<td>102.90</td>
<td>101.30</td>
<td>104.30</td>
</tr>
<tr>
<td>Max</td>
<td>125.90</td>
<td>122.00</td>
<td>144.70</td>
</tr>
</tbody>
</table>

us with the initial term structures for the three countries. The results are displayed in figure 7.1.

Figure 7.1: Yield curves for unmodified data with Nelson Siegel interpolation.

7.1 One-factor Hull-White model implementations

We will start by reviewing the results of models which are combinations of the Hull-White 1 factor model and the B&L model. In section 5 multiple implementations were suggested, we will limit our review to modifications of the linear price, temporary impact form. This restriction is obvious because of the limitations other forms have. Therefore, the last three models in section 5.1 will be implemented and reviewed in
this section. We will first consider the model that follows the following mathematical representation:

\[ P_t = \tilde{P}_t + \Delta_t, \quad t > 0 \]  
\[ \tilde{P}_t = \tilde{P}_{t-1} + (\theta^{(1)}_t - \alpha \tilde{P}_{t-1}) dt + \sigma dW_t \]  
\[ \Delta_t = \left( \theta^{(2)} S_t + \gamma X_t \right) \tilde{P}_t \]  
\[ X_t = \rho X_{t-1} + \eta_t \]

In the above model \( P_t \) now represents the instantaneous forward rate at time \( t \). \( \theta^{(1)} \) is calibrated such that the process fits the initial curve using the method in section 5.3, and we choose \( \theta^{(2)} \) as the (temporary) liquidity premium due to the purchase of \( S_t \) 'interest-rate equivalents' at time \( t \). Note that as said before the interpretation of this model is rather abstract, due to direct investment in the short rate.

There are some technical issues we need to consider before the above model can be implemented in R. First of all, the model includes a \( dt \) term, which should be chosen small enough such that the mean-reversion component does not become too strong. However, choosing \( dt \) small implies that the computation complexity required to complete the simulation grows beyond the power available to us. Note that to simulate enough points to construct a yield curve we would need data for approximately 30 years. Choosing \( dt = 0.1 \) tradingday, which is not very large, would mean that there are 78,000 time points to consider. In our optimization scheme with 20 share blocks to acquire and 160 price and 21 state discretizations to consider that would imply over 5 billion evaluations of our optimality equation. Clearly this is not feasible. Other than being unfeasible, choosing a small time step also implies that there are increasingly many points at which the country can attract money. For comparison purposes, but also for reasons of mimicking reality we have chosen to limit the number of times money can be attracted to once per day. This can however be realized without increasing the \( dt \) value, simply by imposing restraints on the trading moments.

We have decided to limit the simulation to the active trading period, i.e. the working week in which the country requires the funding. Also, we have set \( dt = 1/3\)-tradingday.

Finally, trading is limited to end of day trades. These settings allow us to evaluate different scenarios relatively quickly, which is preferred as we wish to compare methods rather than get optimal results for a specific scenario. Limiting the simulation to the working week means that we require far less simulations, but also that we will not be able to construct yield curves based on our simulation only. We can however append our data with the yield curve data from the dataset, from which we can then construct a new curve. For this method to mathematically hold we should calibrate the parameters...
on the curves that follow from interpolation on the unmodified data. Rather than doing the fully correct calibration, which would be very time consuming, we have chosen to match key parameters such as initial short rate. Limiting our calibration effort this way is mathematically less sound and should not be done when the model is used for pricing, however for a comparative analysis this method suffices. The expected error is small, as the most crucial parameters are either calibrated to the model or chosen conservatively. Furthermore, the $\theta^{(2)}$-term cannot be calibrated to market data, as discussed earlier.

Now that crucial new parameters have been discussed we can review the model results under different liquidity dependency settings. We wish to compare the effects of liquidity dependency and mean-reversion levels under different models. These values will therefore be varied between reasonable levels, after which we will discuss the results. Before going to such situations we first check whether the model works as we expect it under the B&L results. This means that under no state dependency we expect the naive strategy $\bar{S}/T$, limited to the available trading moments. Recall that we consider a country that requires EUR 100M within 5 working days. There are three time steps per day and only at day’s end can trades be made. We set $\gamma = 0$, and obtain the following naive strategy (Note that each package represents 5,000,000 units):

Table 7.2: Naive strategy under first HW1f model.

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

The implementation works as expected, hence we can now proceed towards comparison of the methods. To this end, we will review the effect on the interest rate and yield curve under different models, under similar choice of variables.

Although the model by B&L has been developed to base the trading strategy on market conditions (i.e. the state variable) the models can more easily be compared when the state dependency is minimal. Thus, comparability comes at a price and we should therefore not consider the model in its best use case, but rather its performance in the trivial case. We will however also include a sample in which the state dependency is increased, albeit more for illustration purposes than for actual comparison. Also note that if we choose any other value than $\alpha = 1$ for the mean reversion the mean does not in fact revert to the level of $\theta^{(1)}$. We will consider three different situations, for which the parameters are listed in table 7.3.

Under the Medium/Low parameter setting the first model yields the naive strategy seen above. This is as expected because the low dependency on the state variable makes the
Table 7.3: Choice of variables for comparison of Hull-White model variants.

<table>
<thead>
<tr>
<th></th>
<th>α</th>
<th>(g^{(2)})</th>
<th>γ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medium / Low</td>
<td>1 5 (10^{-9})</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>High / Low</td>
<td>1 1 (10^{-8})</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>Low / High</td>
<td>1 3 (10^{-9})</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>

relative impact of the actual money attraction very large. In such situations the naive strategy is optimal, as we have seen in section 3. The realized interest-rate pattern is what we are actually interested in, which is displayed in table 7.4.

Table 7.4: Strategy and resulting interest rates under Medium/Low parameter choice in first HW1f model.

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Interest rate</td>
<td>3.50</td>
<td>3.46</td>
<td>3.83</td>
<td>3.46</td>
<td>3.83</td>
<td>3.88</td>
<td>3.46</td>
<td>3.83</td>
<td>3.88</td>
<td>3.46</td>
<td>3.83</td>
<td>3.88</td>
<td>3.46</td>
<td>3.83</td>
<td>3.88</td>
</tr>
</tbody>
</table>

Recall from section 5 that we expected the mean-reversion component to be too strong in models where purchases only effect temporary term \(\Delta_t\). Indeed, we see that the interest rate spikes by approximately 0.4% at the purchase moments, and immediately falls back to its previous level in the next time step (plus deviations due to state dependency and error terms). We now increase the liquidity dependency, such that we can review the High/Low scenario. This increases the impact of money attraction even further, hence we expect to see the naive strategy once more. Table 7.5 displays the observed results.

Table 7.5: Strategy and resulting interest rates under High/Low parameter choice in first HW1f model.

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Interest rate</td>
<td>3.50</td>
<td>3.50</td>
<td>4.21</td>
<td>3.50</td>
<td>3.50</td>
<td>4.21</td>
<td>3.54</td>
<td>3.54</td>
<td>4.25</td>
<td>3.54</td>
<td>4.29</td>
<td>3.58</td>
<td>3.54</td>
<td>4.21</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.5 is nearly identical to 7.4, albeit that the spikes are more pertinent. Otherwise, we see the naive strategy and too strong mean reversion as under the Medium/Low setting. Increasing the liquidity dependency to higher levels presents boundary problems, due to the number of discretizations of \(P\). If liquidity dependency goes to such levels it becomes optimal to attract all funds at a single point in time, as the upper bound is met anyway. Therefore it becomes optimal to reach that bound only once rather than five times. This is of course wrong, as in reality there would not be an upper bound. Now, for illustration purposes, let us consider the Low/High scenario, which is arguable more realistic (but allows for poor comparison due to randomness).

In table 7.6 we see that due to the relatively high dependency on market state the sample strategy deviates significantly from the naive strategy. Because the state of the
Table 7.6: Strategy and resulting interest rates under Low/High parameter choice in first HW1f model.

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market State</td>
<td>-0.43</td>
<td>-0.87</td>
<td>-1.73</td>
<td>-2.17</td>
<td>-0.87</td>
<td>-2.60</td>
<td>-1.30</td>
<td>-0.43</td>
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<td>-0.43</td>
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<td>0.43</td>
<td>-0.87</td>
<td>-0.87</td>
<td>-1.30</td>
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<tr>
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<td>9</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Interest rate</td>
<td>3.29</td>
<td>3.08</td>
<td>3.29</td>
<td>2.42</td>
<td>3.08</td>
<td>2.79</td>
<td>2.83</td>
<td>3.29</td>
<td>3.58</td>
<td>3.29</td>
<td>2.62</td>
<td>3.75</td>
<td>3.08</td>
<td>3.08</td>
<td>3.00</td>
</tr>
</tbody>
</table>

Market was low at $t = 3$ and $t = 6$ it was optimal to attract a relatively large portion of the funds early in the time line. The effect of said attraction is relatively small, due to the lower $\theta^{(2)}$ parameter. This example illustrates how the model could in reality help to decide how funds are to be attracted. Also, it should be clear that proper calibration (section 5.3) is crucial for practical purposes.

Yield curves can be constructed using the above method. The relation between the instantaneous forward rate and the price of zero coupon bonds is trivial, as it follows directly from the no arbitrage argument. If we denote by $P(t, T)$ the price of the zero coupon bond from $t$ to $T$ then the forward rate from $t$ to $T$ can be derived by taking the natural logarithm over that period. Once enough forward rates are known a yield curve can be interpolated, using one of the methods in section 5.4.

$$P(t, T) = \mathbb{E}[e^{-\int_t^T r_s ds}] \quad (7.5)$$

$$f(t, T) = \frac{\delta}{\delta T}ln(P(t, T)) \quad (7.6)$$

However, because we have only simulated a short period, full yield-curve construction is not feasible. As noted before we can append our data with that from the original curve. Then we must however note that the liquidity only impacted the instruments very early in the curve. In reality it is more likely that such drastic liquidity crises also influence bonds with longer maturities rather than only those with instantaneous duration. Nevertheless, we will construct the implied yield curve on the Austrian data-set under the Medium/Low and High/Low scenarios. To this end, we have added a one year zero coupon bond, which follows the simulated interest rate. The original curve as well as the implied yield curves under both scenarios are displayed in figure 7.2. During the trading week the curve would take several forms between these two levels, the constructed curves are based on highest rates observed in the period.

As expected, the yield-curve shifts are minimal. We can clearly observe the point for which we have simulated the interest rate. However, as we have only simulated a relatively short period and extrapolated that information to only one interest-rate product the impact on the shape of the curve is minimal. This poses a problem for our comparative analysis that we wish to perform later. Yield-curve based comparison proves difficult for models of this form, unless we increase the simulated period to
one-factor hull-white model implementations

Figure 7.2: Austrian yield curves, unmodified, Medium/Low and High/Low model calibration.

several years. While in theory that is definitely possible, doing so for all scenarios and
models we wish to compare within the available time frame and computational power
is unfeasible. This restriction applies to all variants of the Hull-White model, as those
have a $dt$-term. We can however perform a comparison based on the characteristics
of the interest-rate paths that manifest in these variants.

We will now consider the second of three suggested models that build on the one-factor
Hull-White model:

$$P_t = \tilde{P}_t + \Delta_t, \quad t > 0$$ (7.7)

$$\tilde{P}_t = \tilde{P}_{t-1} + (\theta^{(1)}_t - \alpha P_{t-1}) dt + \sigma dW_t$$ (7.8)

$$\Delta_t = \left(\theta^{(2)}_t S_t + \gamma X_t\right) \tilde{P}_t$$ (7.9)

$$d\theta^{(1)}_t = \bar{d}\theta^{(1)}_t + \theta^{(2)} S_t$$ (7.10)

$$X_t = \rho X_{t-1} + \eta_t$$ (7.11)

The main difference between this model and the previous one is that we now have an $S_t$
term in the mean-reversion component, and thus attracting funds permanently increases
the prevailing interest rate. Recall that expectations are that the permanent increase in
mean reversion is too strong. We will review the model results under the same choice of
variables we have used before. From a modeling perspective there is another significant
difference with the previous model. Since the $\theta^{(1)}$ variable now depends on the chosen
action, it has to be included as a dimension over which we need to optimize. This
means that the number of operations and thus computational complexity increases with
another factor, namely the number of discretizations of the possible $\theta^{(1)}$ values. In the
current setup $\theta^{(1)}$ can only move upwards, thus the lower bound is simply the initial
value. The upper bound is also easily derived: it is the full impact $\theta^{(1)} + \bar{S} * \theta^{(2)}$. We
can now relate the step size to the number of discretizations, such that the rounding
error is minimized. Since each package has an impact of $\theta^{(1)}$ * shareblocksize on the
mean reversion, it would be wise to choose that as the step size. The direct result is
that the number of discretizations should be equal to the number of share blocks, i.e. \( \tilde{S}/\text{shareblocksize} \). In the current setup this value is 20. In other scenarios the value may need to be changed. Examples are scenarios where the relation is no longer linear, or when the boundaries are defined differently.

Under the Medium/Low model setup this model yields the results displayed in table 7.7. The mean-reversion level at time \( t \) is also listed to illustrate how this could work in practice. Note that the calibration is such that after all orders are executed the maximum level of mean reversion is always met. (i.e. the mean-reversion behavior is fully deterministic).

### Table 7.7: Strategy and resulting interest rates under Medium/Low parameter choice in second HW1f model.

<table>
<thead>
<tr>
<th>( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
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<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Mean-reversion Level</td>
<td>3.5</td>
<td>3.5</td>
<td>3.56</td>
<td>3.56</td>
<td>3.67</td>
<td>3.67</td>
<td>3.78</td>
<td>3.78</td>
<td>3.83</td>
<td>3.83</td>
<td>3.83</td>
<td>3.83</td>
<td>3.94</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Interest rate</td>
<td>3.5</td>
<td>3.5</td>
<td>3.83</td>
<td>3.58</td>
<td>3.5</td>
<td>4.08</td>
<td>3.58</td>
<td>3.67</td>
<td>4.17</td>
<td>3.71</td>
<td>4.13</td>
<td>3.83</td>
<td>3.75</td>
<td>4.25</td>
<td></td>
</tr>
</tbody>
</table>

We immediately observe the increasing trend in the interest path: a temporary upward shock when the order is executed and reversion to a higher mean afterwards. As the number of placed orders progress the interest rate climbs to higher levels, more like we would expect than under the model we have considered before. A clear downside of this implementation is that there is no upper boundary on the mean-reversion level (other than the programmatic discretization boundary). A possible extension that could handle this problem is a mean-reversion factor on the mean-reversion component. In reality this variable should be thought of as the time it takes for a crisis to subside, such that all has reverted to its pre-crisis state. Clearly such a mean reversion on the mean reversion should be very slow, in the order of months or possibly years. Although interesting, we will not delve deeper into this subject due to the already time-consuming nature of the solution procedure.

### Table 7.8: Strategy and resulting interest rates under High/Low parameter choice in second HW1f model.

<table>
<thead>
<tr>
<th>( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<th>12</th>
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<th>14</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
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<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Mean-reversion Level</td>
<td>3.5</td>
<td>3.5</td>
<td>3.71</td>
<td>3.71</td>
<td>3.92</td>
<td>3.92</td>
<td>4.13</td>
<td>4.13</td>
<td>4.34</td>
<td>4.34</td>
<td>4.34</td>
<td>4.34</td>
<td>4.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Interest rate</td>
<td>3.5</td>
<td>3.42</td>
<td>3.96</td>
<td>3.63</td>
<td>3.58</td>
<td>4.08</td>
<td>3.67</td>
<td>3.71</td>
<td>4.21</td>
<td>3.83</td>
<td>3.83</td>
<td>4.29</td>
<td>3.92</td>
<td>3.96</td>
<td>4.42</td>
</tr>
</tbody>
</table>

Under the High/Low calibration we see similar results (table 7.8), albeit somewhat more distinct. Here it is more visible that the increased mean-reversion level will be permanent, even after the execution of the order has long passed. Clearly this calls for a solution such as the one mentioned. Otherwise the method behaves as we would have expected after having seen the results under the Medium/Low calibration. For illustration purposes we also include the observed path under Low/High calibration.
in table 7.9. Considering the path below it should now be clear that this calibration is indeed not suitable for comparison purposes. It does however nicely illustrate the practical application of the model.

**Table 7.9:** Strategy and resulting interest rates under Low/High parameter choice in second HW1f model.

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>6</th>
<th>7</th>
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<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market State</td>
<td>-0.43</td>
<td>-2.17</td>
<td>-0.87</td>
<td>0.43</td>
<td>-0.43</td>
<td>-0.43</td>
<td>-1.73</td>
<td>-0.43</td>
<td>0.43</td>
<td>0.43</td>
<td>-0.87</td>
<td>0.43</td>
<td>0.43</td>
<td>0.87</td>
<td></td>
</tr>
<tr>
<td>Strategy</td>
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<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Mean-reversion Level</td>
<td>3.5</td>
<td>3.5</td>
<td>3.56</td>
<td>3.56</td>
<td>3.56</td>
<td>3.63</td>
<td>3.63</td>
<td>3.63</td>
<td>3.66</td>
<td>3.66</td>
<td>3.66</td>
<td>3.75</td>
<td>3.75</td>
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<tr>
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<td>3.46</td>
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<td>3.71</td>
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<td>3.83</td>
<td>3.75</td>
<td>3.92</td>
<td>3.92</td>
<td>4.38</td>
</tr>
</tbody>
</table>

Although the paths we have observed in tables 7.7 - 7.9 are clearly different from the ones we have seen in the previous model, yield-curve comparison will not indicate so. This is because we have looked at the differences between the initial curve and the implied curve at the highest interest-rate level. The developments between those paths are hard to visualize, and due to the subtlety of the differences it is better to describe them numerically. Nevertheless we have still included the curves for integrality. Figure 7.3 displays the implied curve at the maximal observed interest level under the High/Low calibration. Note that this figure is nearly identical to those in figure 7.2.

**Figure 7.3:** Austrian yield curve under second Hull White variant. High/Low parameter choice.

The third variant of the Hull-White 1 factor model has the characteristics as described in equations 7.12 - 7.15. This implementation is expected to be a middle ground between the previous two models. There is a more permanent impact on price than under the first model, but less permanent than under the second model. The mean-reversion level
remains unchanged, but price effects are now on the $\tilde{P}$-level, rather than on the $P$-level.

\begin{align}
\tilde{P}_t &= \tilde{P}_{t-1} + \left( \theta^{(1)}(t) - \alpha \tilde{P}_{t-1} \right) dt + \theta^{(2)} S_t + \sigma dW(t) \\
\Delta_t &= \gamma X_t \tilde{P}_t \\
X_t &= \rho X_{t-1} + \eta_t
\end{align}

An advantage of this implementation over the second implementation is that we no longer require to explicitly model the value of the mean-reversion parameter, which should significantly speed up calculations. We will first review a sample path produced under the Medium/Low model calibration, displayed in table 7.10.

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<th>10</th>
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<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
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<tr>
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<td>3.47</td>
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<td>3.51</td>
<td>3.51</td>
<td>3.64</td>
<td>3.55</td>
<td>3.55</td>
<td>3.60</td>
<td>3.51</td>
<td>3.51</td>
<td>3.60</td>
<td>3.51</td>
<td>3.55</td>
<td>3.64</td>
</tr>
</tbody>
</table>

The above interest-rate path is mostly as expected, but we see that the mean-reversion speed is very high. This would call for smaller time steps, or a constant factor to reduce the speed at which mean-reversion occurs. We will now review the results of the High/Low calibration. In that setup, we do not want the interest rate to revert to its mean in a single time step, rather we want it to occur slower than the trading moments, such that the effect of money attraction are in fact compounding. That is the effect we expect to observe in reality as well.

In table 7.11 we again observe too strong mean reversion, as in the time step immediately after the trading moments the rates have reverted. This calls for the calibration of the $\alpha$-constant in the model, which is indeed practice in standard Hull-White calibration. Currently this value has been left unchanged at 1. To get an idea of how the model could perform once calibrated to market conditions, we also consider the High/Low parameter choice with an additional lower mean-reversion setting. The results are listed in table 7.11 as well. Note that the slow reverting model results in a "bathtub" strategy, opposed to the naive strategy under the near immediate reverting model. Such strategies are expected in situations with external parameters, such as market dependency or in this case mean reversion [2]. This implementation of the one-factor Hull-White model variants is the most promising one as it has a more permanent effect than the first model, and requires far less computational power than the second. Also, for the second model to be fully effective we would require mean reversion on the mean-reversion factor. If however computational complexity is not an issue such a version may prove more valuable than this third implementation, since more parameters
can be calibrated. Before making definitive conclusions the model performance should however be compared in market calibrated scenarios.

Table 7.11: Strategy and resulting interest rates under High/Low parameter choice in third HW1f model.

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
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<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
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<td>Strategy</td>
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<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
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<tr>
<td>Interest rate</td>
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<td>3.51</td>
<td>3.55</td>
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<td>3.48</td>
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<td>3.72</td>
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</tr>
<tr>
<td>Strategy (slow reversion)</td>
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<td>6</td>
<td>0</td>
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<td>2</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>Interest rate (slow reversion)</td>
<td>3.48</td>
<td>3.48</td>
<td>3.77</td>
<td>3.73</td>
<td>3.69</td>
<td>3.77</td>
<td>3.77</td>
<td>3.69</td>
<td>3.77</td>
<td>3.77</td>
<td>3.73</td>
<td>3.90</td>
<td>3.86</td>
<td>3.81</td>
<td>4.072</td>
</tr>
</tbody>
</table>

7.2 Two-factor Hull-White model implementations

We now continue towards two-factor Hull White variants. Recall from section 5.2 that we have two implementations for evaluation. The first implementation follows the following mathematical representation:

\[
dx_t = k_x (\theta_x - x_t) dt + \theta S_t + \sigma_x dW_{1,t}, \quad t > 0
\]

\[
dy_t = k_y (\theta_y - y_t) dt + \sigma_y dW_{2,t}
\]

\[
dW_1 dW_2 = \rho dt
\]

\[
r_t = x_t + y_t
\]

We have decided to choose the parameters such that they could have occurred in reality. Two scenarios are chosen, and on each of these scenarios we will again apply the same liquidity dependency levels as in the previous section. This implies a total of 6 executions will be made per model, from which we will discuss the most interesting results only. Table 7.12 displays the choice of variables under both scenarios. The liquidity dependency parameter settings have also been repeated.

Table 7.12: Choice of variables for comparison of Hull-White model variants.

<table>
<thead>
<tr>
<th></th>
<th>(\theta^{(2)})</th>
<th>(\sigma)</th>
<th>(x)</th>
<th>(y)</th>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medium / Low</td>
<td>5 \ast 10^{-9}</td>
<td>5%</td>
<td>0.5</td>
<td>0.1</td>
<td>0.25</td>
<td>0.1</td>
</tr>
<tr>
<td>High / Low</td>
<td>1 \ast 10^{-8}</td>
<td>5%</td>
<td>\theta_1</td>
<td>2</td>
<td>1.5</td>
<td>\theta_2</td>
</tr>
<tr>
<td>Low / High</td>
<td>3 \ast 10^{-9}</td>
<td>20%</td>
<td>\rho_1</td>
<td>0.5</td>
<td>\rho_2</td>
<td>-0.9</td>
</tr>
</tbody>
</table>

Discretization related variables are not mentioned. They have been set such that the step size is small enough for changes to occur at the smallest execution size. Also, the number of \(x\)-discretizations is larger than \(y\)-discretizations, as much more movement is expected in this component. Now, before evaluation the models according to the above scenarios we first verify whether we retrieve the naive strategy if the model is calibrated without any volatility. If this were not the case we would likely have a programmatic
error in our implementation. Table 7.13 displays the results under $\theta(2) = 5 \times 10^{-9}$ and $\sigma_x, \sigma_y = 0$.

**Table 7.13:** Confirmation of naive strategy under first HW2f model.

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<th>5</th>
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<th>10</th>
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<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
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<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>4</td>
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<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Interest rate</td>
<td>3.48</td>
<td>3.48</td>
<td>3.58</td>
<td>3.56</td>
<td>3.56</td>
<td>3.66</td>
<td>3.65</td>
<td>3.61</td>
<td>3.71</td>
<td>3.69</td>
<td>3.65</td>
<td>3.71</td>
<td>3.69</td>
<td>3.65</td>
<td>3.71</td>
</tr>
</tbody>
</table>

We now consider the Medium/Low liquidity dependency setting. The results are displayed in table 7.14. The results of both scenarios are combined.

**Table 7.14:** Results of Medium/Low model calibrations under first HW2f model.

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
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<tr>
<td>Interest rate (1)</td>
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<td>3.43</td>
<td>3.35</td>
<td>3.36</td>
<td>3.48</td>
<td>3.32</td>
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<td>3.58</td>
<td>3.61</td>
<td>3.54</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Strategy (2)</td>
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<td>0</td>
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<tr>
<td>Interest rate (2)</td>
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<td>3.49</td>
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<td>3.50</td>
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<td>3.55</td>
<td>3.43</td>
<td>3.45</td>
<td>3.54</td>
<td>3.61</td>
<td></td>
</tr>
</tbody>
</table>

In both scenarios we observe the relatively high impact of the variance. The largest interest jumps under the first scenario are not at the purchasing moments, indicating the relatively small impact of the liquidity dependency. This is not fully unexpected, as we have now used the Medium/Low calibration. Under the high model calibration we will consider next we would not expect to observe similar results. Another observation worth mentioning is that the second scenario results deviate more from the naive strategy than the first scenario. This is likely due to the lower mean reversion, which means the relative importance of the variance is increased even more. Similar to high $\gamma$-values we have seen under the HW 1 factor models, this results in more responsive behavior of the model. Again, this indicates that such models can have significant added value in the execution of large orders in volatile situations. We now continue towards the High/Low model calibration, of which the results are listed in table 7.15.

**Table 7.15:** Results of High/Low model calibrations under first HW2f model.

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tr>
<td>Interest rate (1)</td>
<td>3.49</td>
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<td>3.71</td>
<td>3.69</td>
<td>3.52</td>
<td>3.82</td>
<td>3.97</td>
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<td>2</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Interest rate (2)</td>
<td>3.64</td>
<td>3.62</td>
<td>3.76</td>
<td>3.60</td>
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<td>3.60</td>
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<td>3.96</td>
<td>4.03</td>
<td>4.15</td>
<td>4.04</td>
<td>4.21</td>
</tr>
</tbody>
</table>

Under the High/Low parameter calibration we do observe the largest interest spikes precisely at the order moments. This is as expected, as the full impact of the current calibration is $\frac{1}{3.5}$, or nearly 30%. Again we observe more deviation from the naive strategy under the second model calibration. Also the final interest rate at $t = 15$ is higher under the second scenario, both results are explained by the far slower mean reversion. We now consider the model in its most useful use-case, under the Low/High parameter calibration. Again, these results (table 7.16) should not be used
for comparison purposes as there is significant variance. We clearly observe the very high variance, and the fact that the strategy becomes to wait long enough for the rates to drop due to the variance before execution. Under the first scenario this does not happen, and the bulk of the order is executed at $t - 15$. Under the second scenario the rates drop significantly at $t = 5$, and the strategy executes a large portion of the order subsequently. Repetitions of the execution show that the observed behavior does in fact happen under both scenarios (as expected).

Table 7.16: Results of Low/High model calibrations under first HW2f model.

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy (1)</td>
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<td>0</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>11</td>
</tr>
<tr>
<td>Interest rate (1)</td>
<td>3.07</td>
<td>3.50</td>
<td>3.94</td>
<td>3.91</td>
<td>4.21</td>
<td>4.53</td>
<td>5.05</td>
<td>4.69</td>
<td>4.20</td>
<td>4.52</td>
<td>3.76</td>
<td>4.07</td>
<td>3.95</td>
<td>4.10</td>
<td>4.35</td>
</tr>
<tr>
<td>Strategy (2)</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>2</td>
</tr>
<tr>
<td>Interest rate (2)</td>
<td>3.44</td>
<td>3.63</td>
<td>3.74</td>
<td>3.27</td>
<td>3.00</td>
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<td>4.71</td>
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<td>4.44</td>
<td>3.86</td>
<td>3.75</td>
<td>3.39</td>
<td>3.54</td>
</tr>
</tbody>
</table>

We now continue towards to second HW 2 factor model variant. This variant has the indirect liquidity impact: the $\theta S_t$-term is added to the mean-reversion level. Recall the model dynamics from section 5.2:

$$dx_t = k_x(\theta_x - x_t)dt + \sigma_x dW_{1,t}, \quad t > 0 \quad (7.20)$$

$$d\theta_x = d\theta_x + \theta S_t \quad (7.21)$$

$$dy_t = k_y(\theta_y - y_t)dt + \sigma_y dW_{2,t} \quad (7.22)$$

$$dW_1dW_2 = \rho dt \quad (7.23)$$

$$r_t = x_t + y_t \quad (7.24)$$

Before testing this model with the scenarios we have used before we evaluate its behavior under zero volatility calibration. Table 7.17 lists the results. As expected we again retrieve the naive strategy, after the model has been properly calibrated. Calibration of this model is slightly trickier than the previous one. We now again are required to discretize the $\theta$-parameter, which implies significantly more executions of the code. We wish to choose the number of discretizations as low as possible due to the extra execution time, but this needs to be done with care. If the step size is not small enough, increases in mean reversion due to small executions are not applied because of rounding errors. The model will exploit this fact, by executing small orders without influencing the price, and then placing two bulk orders at the last two purchase-moments. A way to avoid most rounding errors is to choose the number of discretizations equal to the number of blocks that are to be acquired. However, modification of the mean-reversion parameters $k_x$ and $k_y$, requires the number of $x$ and $y$-discretizations to be increased accordingly as well.

We now review the model’s performance under the Medium/Low parameter calibration. For this setup the number of $x$ discretizations has been increased to 40, twice the
number of blocks to be acquired. Not doing so results in wrong results, due to the \( k_x = 0.5 \) parameter. For practical purposes, the variance of \( y \) has been set to zero, such that we can decrease the number of \( y \) discretizations to 1, which means that \( y \) is now constant at 1.5. This does not significantly influence the model’s behavior, yet does decrease the computational complexity by a significant amount. The results of the setup are listed in table 7.18.

The results we observe in table 7.18 are highly unexpected, and significantly different from any results we have seen before. To exclude modeling errors, we have varied the parameter choice and discretization sizes. The results remain the same: the largest portion of the order is executed at the final two time steps. Changing the \( k_x \) parameter to values close to or over 1 is the only way to revert to the naive strategy. Herein also lies a possible explanation for the observed behavior: as the order is executed, the mean-reversion level increases. It takes several timesteps for the interest to converge to this new level, i.e. a delayed effect on price. The strategy becomes to hold off purchases until nearly the final moment, such that the effect of the increasing mean reversion is lower than the squared marginal differences that are normally minimized by following a naive strategy. Under the second scenario the results are even more pronounced, as the mean-reversion parameter is even lower at \( k_x = 0.25 \). We now continue to the High/Low parameter calibration.

The results in table 7.19 again show similar behavior. This time, as the liquidity dependency is doubled, the effect of delayed mean reversion is stronger. The direct result is clearly observable when compared to table 7.18: the bulk of the execution is shifted backwards even further than we observed before. The reader may think the
observed results are due to strong volatility. This is however not the case: the volatility
is relatively low at 5%, and repeating the execution yields the exact same strategy in
most cases. We now proceed towards the Low/High model calibration, where volatility
does play a crucial role. The results are listed in table 7.20. Given what we have
observed before the results should not come at a surprise. In the current high volatility
setting it is again attractive to execute a large portion of the order when the price is
low. Combined with the models tendency to execute large orders due to the delayed
price effect this results in the entire order being executed at once, as soon as the rate
drops far enough.

| Table 7.20: Results of High/Low model calibrations under second HW2f model. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| t               | 1               | 2               | 3               | 4               | 5               | 6               | 7               | 8               | 9               | 10              | 11              | 12              | 13              | 14              | 15              |
| Strategy (1)    | 0               | 0               | 20              | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
| Interest rate (1)| 3.92            | 3.37            | 3.80            | 4.09            | 4.72            | 4.43            | 3.88            | 4.26            | 3.99            | 3.44            | 3.52            | 3.60            | 3.86            | 3.86            | 4.33            |
| Strategy (2)    | 0               | 0               | 0               | 0               | 0               | 20              | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
| Interest rate (2)| 3.67            | 4.35            | 4.01            | 3.31            | 2.77            | 2.89            | 3.18            | 3.62            | 3.45            | 3.96            | 4.18            | 4.59            | 4.65            | 3.59            | 3.40            |

This concludes the section on HW 2 factor model variants. We have seen that the
second version of the model does not produce expected output, quite opposite to what
we had originally expected. Next to the unexpected results, this model is also far more
complicated to correctly calibrate without increasing the number of discretizations
beyond a feasible level. The other HW 2 factor model has proved more usable, and
possible has potential for a real-world application. A third option is to combine the
models in a third version, where there is immediate impact like under the first model,
but also the indirect increase in mean-reversion level. Evaluation of such a model
variant is a suggestion for further research.

7.3 Direct simulation of interest product price

We now directly simulate liquidity effects on an interest product, and use the new
pricing information for yield-curve construction. We will mention different calibration
parameters and run the model on all three countries in the data-set. However, as far
from all results are interesting we have decided to limit the graphic illustration to the
most notable results only. We could have chosen to select a single country from the
data-set, so the reader would have gotten a more complete view of the results. By doing
so, we would however introduce the risk of selecting the wrong model, just because it
happens to work well with that specific data.

Directly simulating liquidity effects on a single bond has a disadvantage: bond prices
are highly correlated. Assume five different bonds are available for short-term funding.
Now, it is highly unlikely that in a liquidity crisis a single bond out of those five would
dry up. Far more likely would be to expect price movements on all five bonds, and in case of a full crisis all bonds becoming unavailable. The key question is how we put this relation into the BL model. We have several options:

- Ignore correlations, model price movements on a single bond;
- Have a single bond represent price movements for a range on the yield curve, remove the other points;
- Apply the same percentage-wise price movement on neighboring bonds; and
- Apply the same percentage-wise price movements on neighboring bonds, incorporating some smoothing parameter.

Intuitively the fourth option is the most likely, because we expect the liquidity premium to affect similar products too, albeit somehow smoothed. However, this option imposes more variables and thus more degrees of freedom, making it harder to justify the correctness of the model. We will therefore review all four options, and compare their performance. In section 8 we will present our conclusions.

### 7.3.1 Single-bond price impacts

We will first consider the option where a single bond’s price movements due to trading effects are modeled. We have already expressed our interest in short-term maturity bonds, as those are most heavily influenced during liquidity crises. However, when a single bond is to be selected there is still a range of short-term bonds to select from. We will run the simulation on the first 10 shortest maturities, and review the difference in impact. Sample yield curves will then be constructed using Nelson Siegel interpolation. We will briefly touch upon how realistic each yield curve is, final conclusions will be presented in sections 7.4 and 8. The problem we will consider will be quite small due to the computational complexity of the algorithm and the amount of simulations we wish to compare. The model has been calibrated to the following market conditions:

A country has an immediate financial requirement of EUR 100M. The funds are due within five days, and the country is able to attract funds once a day at blocks of EUR 5,000,000. The state of the market is believed to be positively correlated through \( \rho = 0.5 \), but the impact of the state on the prevailing price is small \( \gamma = 0.001 \). The impact of medium and high liquidity dependency will be compared, by varying the \( \theta \) from \( 1 \times 10^{-9} \) to \( 5 \times 10^{-8} \). These figures, on the scale we are considering, result in a maximum direct cost impact of respectively 0.1% and 5%.
The factors that remain are market volatility and price volatility. These factors should be calibrated on market data in practice but for our comparison purposes it suffices to choose them plausibly. We will use the same $1 - \rho^2$ that BL use for state volatility. Price volatility has been set to 0.02%, in line with normally very constant bond prices.

**Simulations on the bond with the shortest time to maturity.** We will start with the bonds with the shortest maturity. The model by Bertsimas and Lo will be calibrated with the above settings, and we will run Monte-Carlo simulations. We will then select a single path from the simulation and use the price path that follows from that specific execution plan to construct the yield curves that would have been empirically seen during the five trading days. We will first review the case where liquidity impact is medium, at $\theta = 1 \times 10^{-9}$.

The observed sample strategy is listed in table 7.21. Due to the low impact of the state variable we see, as we should expect, the optimal strategy is equal to the naive strategy. The maximum direct impact when all EUR 100,000,000 would have been attracted was a price increase of 0.1%. The price path of the sample strategy shows that as expected, the realized prices are kept between 100% and that level. This is not a very interesting calibration of the parameters, as it results in the trivial naive strategy. Nevertheless, it is useful to use this benchmark in the yield-curve construction, and see how such a price movement on a single bond would impact the yield curve. The state space is discretized into 21 steps, thus state 11 corresponds with a neutral market. Note that there is an inversion between the ‘Price’ and ‘Price of Bond’ columns. An increase in price now would imply the value of the final cash flow increasing, and thus the bank being able to attract more money by writing the same bond. The inverse is true, hence the inversion: the current value of the bond is divided by the price increase. This yields the desired effect, yields on the bonds under influence increase as the values drop.

<table>
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<th>France</th>
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<td>9</td>
<td>99.95202</td>
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<td>99.82009</td>
</tr>
</tbody>
</table>

Under the current calibration the price effects are relatively small, and we would not expect much movement on the yield curve. Figure 7.4 displays the yield curve of French bonds under the original circumstances on the left, and the recomputed curve under
the strongest price movement, i.e. the one at \( t = 5 \) on the right. We can clearly see the first point moving upwards, but effects on the fitted curve are minimal. Curves of German and Austrian bonds have been omitted for brevity, as they show similar minimal movements.

![Figure 7.4: Comparison of French yield curves under weak model calibration.](image)

We now repeat the procedure, this time under stronger price dependency. We have set \( \theta = 5 \times 10^{-8} \), which corresponds to full impact of 5\%. Other parameters remain the same, and thus as expected we see the naive strategy reappear. Due to the larger price movement the discretization size needs to be increased from 1/200 to 1/50 to avoid problems arising from the boundary. Failing to do so yields unexpected results, as the price approaches the upper boundary of the discretization space. We have seen similar issues in section 3.2.4, where the naive strategy no longer emerges if prices rise too high. Suffice it to say that if the liquidity impact is increased significantly the price space must be widened accordingly. This can be done by either increasing the number of discretization steps or by decreasing the step size.

![Table 7.22: Sample price path under proposed strong parameters and effects on sample bond data-set.](image)

Again, we construct the yield curve under the modified prices and compare it to the original yield curve. In figure 7.5 we clearly see that the yield curve is far steeper due to the replacement of the first data point. We also observe one of the problems that rises from single point modification: the yield on the first point has risen to a level where it is highly unlikely to be correct, and the yield curve significantly deviates from neighboring points. This is reflected in the RMSE, which has risen from 0.04\% in the unmodified scenario to 1.4\% in the curve at \( t = 5 \).
Simulations on other short maturity bonds  Rather than selecting the first bond we can also choose to select any other bond with short maturity. For the purpose of demonstrating why single-bond price impact is not a wise choice the fifth bond in the French package has proven to be useful. Although yield-curve shifts can be observed, the data point that is being modified would be considered an outlier in any reasonable statistical analysis. Figure 7.6 displays the yield-curve shift that would result if the price movements in table 7.22 would have applied on the fifth bond. Also, the extreme yield on the fifth bond is clearly visible. Such a scenario is unlikely to be the best approximation of the effects of a liquidity crisis.

We may now conclude that liquidity effects on single bonds are an unreasonable way to simulate reality. In subsequent subsections we will discuss the other three approaches mentioned earlier, following a similar approach.

7.3.2  Single price effects with neighboring points removed

One way to cope with the issue of multiple short-term bonds in the portfolio is to simply remove the neighboring bonds. The remaining bond should then be thought of as representative for the whole range of short-term bonds. We could attempt to make the remaining bond a weighted average of the duration of all neighboring bonds that are
being removed, which is arguably the most correct approach. This approach is however very time consuming, and does not necessarily yield desired results. Instead, we will simply remove the neighboring bonds and simulate on the remaining one. The number of bonds that will be removed and the representative bond will be varied. Should we find this approach to yield more realistic curves than methods 1, 3 or 4, we can still investigate constructed duration weighted bonds.

Our initial expectation is that this approach is rather crude, and will yield extreme movement in the curves due to missing data points. Especially in cases where the portfolio is smaller the removal of points may cause the shape of the yield curve to shift more radically than we aim for.

To evaluate this approach we will remove up to 10 neighboring bonds, and select one of the first 10 bonds as the representative bond. To avoid having to investigate every possible combination of those two parameters we have chosen to limit our simulations to cases where 3, 5 and 10 bonds are removed. We will then investigate cases where the first, middle, and last bond in the removed series are representative, that bond will be subject to price movements.

Using the last bond in the neighboring series as representative bond proves to be unsuccessful. Doing so results in very limited yield-curve movement, and more importantly the complete removal of shorter term maturities from the curve. A single example displayed in figure 7.7 should be enough to understand the implications of this method. Note the short x-axis, a direct result of removing all points prior to the representative bond.

Figure 7.7: 20-Neighbor removal method with 20th bond as representative.
Price movements have limited effect on longer maturity bonds, because as the difference is discounted over more periods the impact on yield decreases. Therefore, the net result is that basically the yield curve is cut short, and no longer provides any information on short-term maturities. We will drop this approach from this point onward.

Using the first bond as representative yields very extreme yield-curve behavior. As we saw in the first method price movements on the first bond have strong effects on yield, and thus on the curve. Removing all neighboring points after the first point up to a certain number amplifies this effect, as all countering forces are now removed. As long as price movements remain small we still observe plausible curves. However, increasing liquidity dependency to medium levels ($\theta = 5 \times 10^{-8}$) results in curves that could never present in reality. Figure 7.8 displays the effect of four liquidity dependency levels on the French curve. Clearly only the low dependency level is even remotely viable.

A brief review of the simulations was enough to conclude that first and last points are not suitable candidate representative bonds. Results for the middle bond are more promising, although still prove to be very variable. Our expectation of crudeness appears to be correct: again a small liquidity dependency yield drastic curve shifts. However, if we choose the dependency to be low to medium we do find some curves...
that might occur in reality. This is an indication that proper choice of variables and calibration could lead to actual insight in the effects of liquidity shortage on the curve. Figure 7.9 displays two German curves which show how limited dependency can offset the curve in a way that could happen in a liquidity crisis.

![German curves](image)

**Figure 7.9:** Neighbor removal method on German bonds, respectively 20 and 30 bonds removed. Middle bond representative. $\theta = 5 \times 10^{-8}$.

At this point we do not wish to definitively discard this approach, even though most of the results are too crude and extreme. However, proper choice of variables and calibration may counter this problem, which could lead to more results as seen in figure 7.9. Also, we still have not considered the option of a weighted average representative bond, which could also refine the results.

### 7.3.3 Similar price effects on neighboring bonds

The third possibility we consider is applying the price effect on several neighboring bonds, starting at the first one. We expect a ‘jump’ effect as the effect suddenly stops. Also, we expect the price effect to be larger on bonds with shorter maturities as the yield is computed over a shorter period. A price-change of 1% on a bond with overnight maturity has a far smaller effect on yield than an equal change on a bond that matures in three months.

For consistency reasons, we apply the same price path we found earlier. The price movements are then applied on neighboring bonds. One question that remains is how many neighboring bonds should be affected, we have chosen to review 3, 5 and 10 neighbors. Table 7.23 displays the prices we find when we apply this scheme on the French bond package. We will not consider German and Austrian bonds for reasons of brevity, as they show similar patterns.
Table 7.23: Sample price path under proposed weak parameters and effects on French bond data-set.

<table>
<thead>
<tr>
<th>t</th>
<th>Strategy</th>
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<th>State</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>99.84504</td>
<td>100.18495</td>
<td>99.54511</td>
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<tr>
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<td>4</td>
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<td>11</td>
<td>99.85502</td>
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</tr>
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<td>4</td>
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<td>13</td>
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<td>99.56502</td>
</tr>
<tr>
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<td>100.040</td>
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<td>99.53019</td>
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<tr>
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<td>9</td>
<td>99.82009</td>
<td>100.15992</td>
<td>99.52024</td>
</tr>
</tbody>
</table>

The price effects in table 7.23 are those that follow from the weak calibration. We expect the yield-curve movements to be minimal, although stronger than under the single-bond impact scenario. Also, we expect less ‘definite outliers’, as there are three movements with a dampening effect. Figure 7.10 displays the resulting yield curve compared to the original curve.

On first glance this method does outperform the single-bond modification approach, although the movements in figure 7.10 are too small for a proper evaluation. We will now consider the impact of strong model calibration in scenarios with 3, 5 and 10 neighboring bonds being affected. Figure 7.11 displays the original curve, followed by the curves resulting from the modification.

As expected we see the dampening effect described earlier, where price effects on bonds with longer maturities have less effect on bond yield. The expected ‘jump’ is best visible in the latest plot, where the yield on bond 10 differs significantly, possibly unrealistically from the yield on bond 11. As a side result of the dampening it seems that the fit is better as the number of affected neighbors is increased. The RMSE figure only partially confirms this fact. The RMSE values for the four scenarios are (respectively) 0.0468%, 0.563%, 0.487% and 0.628%. The 5-neighbor option therefore has the best fit, although still significantly worse than the original scenario. However, empirically considering the plots we may conclude that the 10-neighbor approach is in fact the best choice, due to smaller gaps between the data points.
Finally we may consider the situation where the effect is widened to the first 20 bonds. We expect the price jump between 20 and 21 to be clearly visible, which would suggest the use of a smoothing parameter. However, we do wish to review this option without smoothing to compare it to the performance of the 10-neighbor option. Figure 7.12 displays the 10- and 20-neighbor options on the left and right respectively.

We see that the price jump between 20 and 21 is indeed clearly visible. Also, as the maturity increases the price effect on the yield becomes relatively small. Where an 1.04% increase in price has a drastic effect on the first bonds, the effect on bonds 10-20 is far less. This may indicate that the current method is only able to simulate effects on very short-term bonds, and thus full yield-curve inversion is unlikely to occur. This does however depend on several other factors, such as the total number of bonds in the portfolio, the original shape of the yield curve and the number of medium to high maturity bonds.
7.3.4 Dampening price effect on neighboring bonds

The fourth variant we will consider is similar to the third. The sole difference is that a smoothing parameter will be used to possibly remove the price jump we have seen earlier. The following variables need to be chosen:

- The bond that has 100% effect, and is most sensitive to liquidity effects.
- Number of neighboring bonds that are affected.
- Smoothing method.

For reasons of simplicity we will consider a linear decrease of the price impact across all neighboring bonds. This means that if 4 neighbors are affected the impact will be 100-80-60-40-20 in both directions. Other smoothing methods can be considered if this proves insufficient. In line with earlier methods we will start by letting the shortest maturity have the 100% price effect. We will however also consider others, which would empirically mean that 3- or 6-month bonds are affected more severely in crises due to higher demand. We have some expectations for this approach:

- Jumps in the yield curve can be fully removed if variables are chosen accordingly.
- The effect on the yield curve will be smaller than under previous methods.
- Increasing the liquidity dependency to counter the limited effect on yield will result in too high yields on the first bonds. Shifting the 100% impact bond may be a solution to this issue, enabling broader yield-curve shifts.
- RMSE figures will be lower when compared to similar calibration under method 3.

Since we want to be able to compare methods 1-4, a reasonable choice is to use the same number of neighbors we chose earlier. This means that we will first consider 3, 5, and 10 bonds being affected. We start by letting the first bond have 100% impact, this way we can easily compare methods 3 and 4. If as expected the effect on the yield curve is smaller we will also consider higher liquidity dependency.

Let us first review the result of $\theta = 5 \times 10^{-8}$, affecting three neighbors. The smoothing will be as 100-66-33%. Table 7.24 displays the prices that follow from this parameter setting.

In figure 7.13 the 3-bond impact with and without smoothing parameters are compared. Although the difference is small, we can clearly see the smoothed version has a smaller
Table 7.24: Sample price path under proposed strong parameters and effects on French bond data-set.

<table>
<thead>
<tr>
<th>t</th>
<th>Strategy</th>
<th>Price</th>
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<th>2</th>
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</tr>
<tr>
<td>3</td>
<td>4</td>
<td>101.02</td>
<td>12</td>
<td>98.86161</td>
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<td>99.22602</td>
</tr>
</tbody>
</table>

effect on yields, and allows for the curve to fit the data points better. Note the effect on the yield on the third bond is around 4.5% opposed to 6%, and on the second bond around 6.3% opposed to 8%.

Figure 7.13: Comparison of 3-bond impact with and without smoothing effect on French bonds.

The smoothing has increasingly more value as the number of neighbors increase. Due to the relative small impact on yield the price jump became more clear when we were considering 5 and 10 (and even 20) bonds under effect. Figure 7.14 compares the 5- and 10-bond scenarios, the non smoothed versions are shown on the left.

Still, the smoothed versions are unlikely the best way to model liquidity crises on interest products because we still see too strong influence on short-term maturities, and too little impact on longer maturities. Increasing liquidity dependency will increase the yield shifts on medium length maturities, but will likely increase short-term yield to unrealistic levels. As mentioned earlier, a possible way around this issue is to shift the full impact bond. We will now review such changes, and thus need to vary an additional variable: which bond is the full impact bond. We will calibrate the scenario in two ways: such that the first bond still has at least 50% impact, and such that the first bond has 10% impact. Both methods fully define which bond is the full impact bond (under the same linear smoothing).

Shifting the full impact bond does have a downside. Next to liquidity dependency, smoothing methodology, and the number of neighboring bonds affected we now introduce
an additional variable that is very hard to calibrate or justify. This must be considered when we later compare this indirect approach to the more direct approaches we discussed earlier.

Because of the high number of variables we have chosen to only demonstrate interesting yield-curve shifts that can be realized when this methodology is followed. Different values of the before mentioned four introduced parameters were examined. Other variables (specific to the BL model) have not been varied.

The Austrian bond set shows the most promising results. This bond set has relatively few long-term bonds, hence the impact of shifts on short-term bonds on the overall yield curve is relatively small. As such, yield-curve inversion is realized earlier than in the French and German datasets. The original Austrian yield curve is a typical convex increasing curve, displayed in figure 7.15.

As we have seen before, the French yield curve can be inverted, but this results in extreme yields on especially the first few bonds in the dataset. Now, consider the Austrian dataset with medium impact at $\theta = 5 \times 10^{-8}$. Figure 7.16 displays the resulting yield curve when the 10th bond is the full impact bond, and the first bond is at least at 50% impact.

Figure 7.16 may not seem much different from 7.15 at first glance, but this could be a typical example of a flattening curve which traditionally precedes an economic decline. Increasing the liquidity dependency to higher levels (i.e. $\theta = 15 \times 10^{-8}$ in the figure below) shows full yield-curve inversion without spiking the yield on the first bond to
unrealistic levels. Figure 7.17 displays the resulting yield curves when the 10th and 15th bond are full impact bonds. Interestingly, setting the first bond to at least 50% impact shows far more promising results than the lower 10% variant. Both plots in figure 7.17 are under the 50% setting.

More inverted Austrian curves can be realized by increasing dependency to $\theta = 25 \cdot 10^{-8}$. Doing so shows curves that could present in reality when investors have far higher expectations for short-term interest rates than for long-term ones, i.e. when a economic crisis is imminent. Similar results can be obtained with the French and German data-sets. Figure 7.18 displays four sample results which could arguably be seen in practice, although the steepness is extreme. During the simulations we have seen far more unrealistic results in the French and German data-sets than under the Austrian
Figure 7.17: Austrian yield curves under strong liquidity impact, 10th and 15th full impact bond.

data-set. This suggests that if this model is to be used in practice, it should be done with care and thoughtful historic calibration. Only then can the results be an indication of the possible implications of financial requirements on the curve. This concludes the section on direct simulation on interest products. In the next section we will compare all discussed methods, and discuss which method deserves our preference.

Figure 7.18: Sample inverted French and German yield curves.
7.4 Conclusion

We conclude this section with a short overview of our observations and present our conclusions on which models we believe are most promising. In total nine different model suggestions were made, of which five Hull-White variants and four direct approaches. We noted that the Hull White variants are most useful in short-term funding requirements, while the direct simulations are more useful in situations where funding is required for the medium term. This distinction was made because the direct simulations have a more drastic effect on the shape of the yield curve, while Hull White variants have very limited effect on the curve. We will therefore present our conclusions on both situations, and recommend a model for both the short-term funding situations as well as the medium term.

**Short-term funding requirements** are those situations where funding is required only for several weeks to at most months. In such situations there is often no deeper underlying problem, but simply a sudden liquidity mismatch that needs to be solved temporarily. Such situations can however cause liquidity premiums if the market is thin, and the requirement is large and imminent. In such situations one of the models of the Hull-White family is likely the best approach. We have thoroughly reviewed all models in sections 7.1 - 7.2, and can conclude that several candidates are not the correct approach. The third variant of the one-factor HW models and the first variant of the two-factor HW models were the best in their categories. The mathematical representations for those two models were as follows:

\[
egin{align*}
P_t &= \tilde{P}_t + \Delta_t, \\
\tilde{P}_t &= \tilde{P}_{t-1} + \left(\theta^{(1)}(t) - \alpha \tilde{P}_{t-1}\right) dt + \theta S_t + \sigma dW(t) \\
\Delta_t &= \gamma X_t \tilde{P}_t \\
X_t &= \rho X_{t-1} + \eta, \quad t > 0
\end{align*}
\]

Several strong points of this one-factor model over other one-factor models are the following.

1. More permanent effect on interest rate than other models considered. This opposed to only a temporary effect in models where mean reversion was instant.
2. Not as computationally complex as models where the $\theta$-term is discretized. This means that with similar computational power larger problems can be solved, or smaller problems can be solved more accurately.

We did however note that if computational time is not an issue at all (which is arguably never the case) then the third model in section 5.1 may be more appropriate.

Strong points of the first two-factor model over the other two-factor model we evaluated in section 7.2 are the following:

1. The model produces strategies that we would expect to be optimal given what we have observed in literature and other implementations, which the second model does not.

2. The model is relatively easy to calibrate, as there is no troublesome discretization parameter calibration.

3. The model is less computationally complex than the other two-factor model.

Now, what remains to be done is to choose the best candidate from the above two models. We argued in section 5.1 that HW 1 factor models are unlikely to be the best choice because of their limitation to parallel shifts on the curve. However, because these models are only to be applied on short-term problems that limitation is considered unimportant. The two-factor model variant has no other advantages than the fact that forward rates are not 100% correlated. The one-factor model does however have two advantages. First, it is far more easy to calibrate the HW 1 factor model to market conditions, in fact easy enough to be described shortly in this thesis in section 5.3. Calibration of the two-factor model is not as trivial, but can certainly be done. If the financial benefits are significant the problem of calibration should not be decisive, as calibration is only a one time task that can be automated afterwards.

The second advantage of the one-factor model is the inclusion of a state variable. This means that the model is able to react to market conditions other than the interest rate, such as the S&P 500 that Bertsimas & Lo use in their original paper. This additional market term can prove valuable in decision making, especially when there is reason to believe the optimal strategy is correlated with such a global market variable. At this point we would recommend the use of both models in parallel, and depending on the situation determine which model is more applicable. Either one that includes market conditions, or one that does not produce correlated forward rates. Ideally, the produced strategies would not deviate much from one another, indicating that both models work fairly well.
Medium-term funding requirements are situations where the funding gap persists over a longer period of several months to years. Such a funding gap in the proportions we are considering is often an indication of deeper, more fundamental problems. It is therefore unlikely that such situations would occur as isolated liquidity problems, without Credit Value Adjustments or other re-pricing methods. However, for the purpose of this research we have only considered isolated scenarios and for such cases we have proposed the method of direct simulations on interest-rate products. We reviewed the following four methods in section 7.3:

1. Ignore correlations, model price movements on a single bond;
2. Have a single bond represent price movements for a range on the yield curve, remove the other points;
3. Apply the same percentage-wise price movement on neighboring bonds; and
4. Apply the same percentage-wise price movements on neighboring bonds, incorporating some smoothing parameter.

Beforehand the expectation was that the fourth method would clearly be the best option, because of its smoothed application. Indeed, the first and second methods produced far more extreme spikes in the yield curve than the latter two methods. We therefore definitively discommend the use of methods one and two. The third method did however produce positively surprising results, as applying the same percentage-wise price movement appeared to have a smoothing effect as well. This was explained by considering the implication of this method: applying the same price shift on products with different maturities means that for products with longer maturities the shift is discounted over more years, thus making the relative importance of the shift smaller. This causes a smoothing effect similar to the one used in the fourth method. In fact, the fourth method has two smoothing effects, which is the reason for the odd bumps that occur in some of the curves. Both methods can be used in practice, and the extra degree of freedom in the fourth method can then be considered superfluous if method three is already sufficient. However, method four is better able to produce shifts on the curve on points with slightly higher maturity, such as we have seen in figures 7.17 and 7.18. It is for this reason that we believe that this method is the best approach, but only if it used with careful calibration. For the moment calibration parameters are still hard to justify, making the additional degree of freedom possibly troublesome. Further experiments on the justification of the smoothing method and parameter are required before choosing a definitive calibration.

Until then we recommend the use of both methods in parallel. This will provide an expected spread of the curve, which can be used as an approximation of the expected
impact. Using multiple models in this case has another advantage: it is very likely that the results will not be equal, and thus there will be significant spread. This indicates that there is high uncertainty in the provided curve, a fact that could have been ignored if only a single model would have been used.
Section 8

Conclusions

In this thesis we set out to construct a model which can be applied to interest-rate products, such that the liquidity premium can be quantified. The actual research question as mentioned in section 2 was as follows:

*How can we relate, model and quantify liquidity risk on interest-rate products?*

We have posed five sub-questions, which combined form an answer to the central question. Our final conclusions will be presented in this section, in the form of answers to the sub-questions we posed in section 2.

We first set out to study existing liquidity models that are currently being applied in practice. We have thoroughly discussed the liquidity model by Bertsimas & Lo, which has functioned as the basis for the liquidity component in all the models we have discussed. Other models have also been studied, not the least of which the Multiplicative Error Measurement model by Engle and Fleming [3]. The model by B&L was however considered superior to other models for our purposes. We have not applied the full possibilities of the model however, as we have limited our implementations to single-state scenarios. The model does also lend itself for applications in multidimensional scenarios, which could prove valuable in future studies. Also, we have used the basic version of the model, which has since then been extended by other researchers. This choice was made for the sake of simplicity, as extensions complicate the model significantly while only improving it slightly or in specific situations. Future studies may show that extended versions of the model prove more valuable for our purposes as well.

After studying liquidity problems we studied interest-rate models. We have restricted our study to the most used models, namely the Hull-White one and two-factor models and the Libor Market Model. Without doubt the LMM-model is the most mathematically sound implementation of an interest model, its computational complexity was however
considered too high for further manipulation. We have therefore restricted the interest models that are applied to variants of the Hull-White model, which are discussed in section 5. We concluded that section with the notion that the one-factor family of models is unlikely to be the best choice, due to a strong limitation: 100% correlated forward rates. However, we found later that model variants that follow the HW approach are only applicable in short-term scenarios, as it is unfeasible to construct longer term yield curves with those models due to the computational complexity exploding state spaces present. Considering that limitation on the application of the model the problem of correlated forward rates becomes less significant.

The third question we posed involved the actual modeling. We discussed our views on how both families of models can be combined. We presented six variants of Hull White one-factor models in section 5.1, of which the most promising three were reviewed in section 7.1. Then, in section 5.2 we presented two variants of the Hull White 2 factor model, which have both been reviewed in section 7.2. Finally, we considered four methods of the ‘direct simulation on interest products’ approach, all of which were evaluated extensively in section 7.3.

The fourth question we posed considered the behavior of yield curves in practice. We extensively described the behavior of curves in section 6. We have seen that preceding economic crises, curves tend to flatten out. Curves then become inverted as the crisis nears and once the economy is recovering return to their original shape, albeit slightly steeper. We have judged the models we considered based on this knowledge where applicable, but also explained that we expected more extreme results than would typically be observed in reality. This is because we are considering implied curves under the full impact precisely at the execution moments. At those moments the temporary price impact is also included, which amplifies the shift. This temporary effect is expected to stabilize shortly after being introduced, because of the markets tendency to flatten out any arbitrage opportunities. The persisting effect is somewhere in between the implied curve and the original curve. That being said, we have only used yield curve based verification for the medium-term funding problems, as the short-term problems have little to no impact on the curve. The short-term problems have been compared by considering the implied paths the interest rate would have followed under each proposed optimal strategy. We used this information to compare the different families of models, and presented our conclusions in section 7.4. Our final recommendation consists of two Hull-White model variants for short-term funding requirements, and two direct simulation methods for medium-term funding requirements. Once calibration parameters have been studied more extensively the number of direct simulation methods should however reduce to one.
Finally we posed the calibration question, i.e. is it possible to calibrate our models to real world situations. We discussed calibration of the Hull White family of models in section 5.3, where we proposed an iterative learning approach for both the one and two-factor model variants. The basis for this learning approach is the default Hull White model calibration, of which we described only the one-factor variant. A reference to the appropriate literature has been provided for the two-factor variant.

Calibration of the direct simulation approach is less trivial, as it requires rerunning Nelson-Siegel interpolation each time the parameters are changed. Therefore it is not easy to find the parameters that correspond to the Nelson-Siegel interpolation leading to the observed yield-curve shift. Adding more variables, such as under the fourth model further complicates the matter. Nevertheless, the approach would be similar to the learning step in the Hull White variant calibration, and has been included in Appendix D.

Now, to answer the central question we can state the following: We can relate model and quantify liquidity risk on interest-rate products, but to do so with a reasonable degree of accuracy the distinction between short and medium-term funding problems must be made. Given that distinction we have provided two models for the short-term problems, which we recommend be used in parallel. We have also provided two variants of the direct approach which should be used in medium-term problems. We expect the fourth model to perform better, but it is also harder to calibrate properly. Again we recommend parallel application, at least until more insight in the calibration of the fourth model is gained. Additionally using both models in parallel will provide insight in accuracy because the spread between the curves can be observed.
Appendix A

Background

This thesis has been written as a master graduation project for the VU University of Amsterdam, during a six month internship with Ernst & Young. In this section we will give a more exhaustive introduction into the broad topic of economic scenario generation, which was later narrowed down to liquidity risk on interest products. Throughout this section we will provide information on risk sources and mitigation applications, this information has all been sourced from [8].

Ernst & Young is a globally operating company, which provides (amongst others) financial advice to clients in many different sectors. This research is done for the financial risk management department (FSRisk) of EY’s Advisory Services. One of FSR’s desires is to offer Economic Scenario generation as a service to its clients.

Economic scenario generation (ESG) consists of mathematical models to simulate possible paths the economy can follow. Once enough paths have been sampled an advice can be provided based on most likely outcomes. Economic scenario generation is a very broad field, which has been narrowed down to the subject of liquidity risk on interest products in the introduction. In this section we will give an overview of other applications of economic scenario generators, and show how and why this subtopic was selected as central research question. A common consensus in modeling is that it is preferred to have multiple models, each serving a specific task, over a single model that models the entire reality. In the context of Economic scenario generation this means that for different forms of risk different models are used, simply because it is near impossible to simulate the entire economy and its inter-dependencies in one model. We will first explore different forms of risk, also known as risk drivers, EY’s primary clients are commonly exposed to. The primary clients of the FSRisk department are mostly large banks and insurance companies.
In the remainder of this section we will review all risk drivers, as well as possible applications ESG-models can offer to mitigate those Risk Drivers. First we present the list of criteria based on which we will decide which sources of risk will be included in our model. In the subsequent section we will present the central research question that has followed from this first literature review.

A.1 Risk driver selection criteria

A list of criteria has been generated, based on which the area of interest has been selected, within the context of ESG. To ensure that this area of interest is in line with EY’s desired outcome this has been done together with colleagues and supervisors. Brainstorming and literature review have together resulted in the following list.

- Risk driver is highly applicable in banking industry;
- Relevant models exist in literature;
- Available models are able to accurately quantify the Risk Driver;
- Model is within the mathematical reach for this thesis;
- Model can be calibrated on market data once implemented; and
- The combination of selected Risk Drivers can be modeled within a 6 month time frame.

A.2 Risk drivers

Risk drivers have been studied extensively in recent history. 'Risk' has been defined as 'The volatility of returns leading to unexpected losses'[7]. Risk drivers are defined as sources from which such unexpected losses may originate. Risk can easily be underestimated, which has led to defaults of some renowned companies in recent history. It is for this reason that the Basel committee has set regulatory guidelines for the banking industry, through which risk management is enforced.

Crouhy et al. identify eight major categories of risk drivers, which will be briefly discussed here. Subcategories exist, most of which can be modeled in multiple ways. We will use the criteria mentioned earlier to select subcategories and accompanying models for our ESG. The following main categories are identified:
• Market Risk;
• Credit Risk;
• Liquidity Risk;
• Operational Risk;
• Legal and Regulatory Risk;
• Business Risk;
• Strategic Risk; and
• Reputation Risk.

A.2.1 Market risk

is often defined as the possibility of losses due to factors that affect the overall per-
formance of the market. Possible examples of market risk are natural disasters, political
turmoil and interest rate risk. Although these risks cannot all be diversified away, it is
possible to hedge against different forms of market risk. Many different models exist to
realize such hedged portfolios, as we will see later. Several sub types of market risk
may be identified, which will be discussed in the following subsections.

**Interest rate risk** arises when the risk free interest rate changes over time. This
typically impacts the value of an investment, as climbing interest rates decrease the
value of future cash flows. Interest rate risk can occur in different ways, the most
apparent one being absolute changes in interest rate level. Other ways are changes
in the yield curve, or changes in the spread between two rates. All such changes can
heavily impact the value of an investment. However, in most situations it is possible
to invest in a security which is inversely affected by interest rate changes, and as such
diversify away the interest rate risk. Other countermeasures include hedges, such as
interest rate swaps.

**Equity price risk**, or simply equity risk, is the risk that arises from volatility in
stock prices. Specific (or idiosyncratic) risk is firm specific Equity Risk, which can
be diversified away by choosing a smart portfolio. Systemic risk on the other hand
is the proportion of Equity risk that cannot be diversified away. It consists of events
that effect the market as a whole, such as major political events or natural disaster.
Investors will generally attempt to choose portfolios such that they minimize the total
idiosyncratic risk they are exposed to, and as such maximize their expected return per unit of risk.

**Foreign exchange risk** (FX risk) occurs when a party willingly or unwillingly is exposed to an unhedged position in a foreign currency. This can occur because they invest in FX derivatives, but also because trading is done in EUR while actual operations are done in EUR. Leaving such positions unhedged can have significant, undesired effects on returns. FX Risk can be countered by investing in numerous FX derivatives such as FX options (also known as Forex options).

**Commodity price risk** is the risk of price fluctuations in production inputs negatively effecting a company’s operations. Commodity markets are known to be more volatile than for instance FX or interest markets, and can thus expose companies to significant risk. Commodities often have few, powerful suppliers, that have strong influence on the market price. Other reasons for increased volatility include fluctuations in the depth of the market, and natural disasters. To ensure a steady profit margin large companies often hedge their commodity requirements.

### A.2.2 Credit risk

occurs when the counter party in a contract is not able to meet its financial obligations. Default is the most extreme case of credit risk, but other scenarios can be thought of where counter parties are unable to fulfill their financial obligations in full. A simple example is when a rating agency downgrades the counter party, thus decreasing its financial strength and possibly its ability to pay its dues in full.

The amount of money lost in case of default is commonly referred to as Loss Given Default (LGD), sometimes specified as a percentage. The Probability of Default (PD) is a percentage estimate of the likelihood of default of a counter party. These figures are notoriously hard to estimate, as they occur in distressed scenarios for which relatively little information is available. PD * LGD is often taken as a risk measure for credit risk.

### A.2.3 Liquidity risk

Liquidity Risk occurs when a company is unable to retrieve the market value for an asset (Asset Liquidity Risk) or when it is forced to pay more than the fair market price for funds (Funding Liquidity Risk).
Asset Liquidity Risk often occurs in distressed market situations, when a party needs to dump a large portion of one of its assets on the market. If the counter part of the market is not willing to buy the offered asset at the normal market price (i.e. the market is illiquid), the party may be forced to sell the assets below its normal market value. Four properties influence asset liquidity in an otherwise stable market.

- Microstructure;
  Depending on the asset type the market can be usually deep or thin: i.e. interest rate options usually have deep markets while OTC products have more thin markets.
- Complexity;
  Complex derivatives are typically less liquid than simple derivative products.
- Substitutability;
  Products for which substitutes exist are more liquid than those without substitute.
- Urgency;
  Shorter time frames generally increase effects of liquidity on the bid-ask spread.

Liquidity risk has led to tremendous losses in recent history, especially when a party owns a large portion of the total available assets of a distressed counter party.

Funding Liquidity Risk basically means a company is not able to fulfill its short-term financial obligations because of a shortage of liquid funds. Note that funding illiquidity is very different, although often hard to distinguish, from insolvency. Insolvency means the company’s Assets have lower total value than its liabilities, and as such the company is bankrupt. Funding illiquidity simply means there is a mismatch between the company’s cash position and short-term liabilities, but the company is otherwise healthy. Funding illiquidity can thus be managed by holding cash or equivalents. Of course, holding such positions can be costly as they do not actively generate money. In the banking industry one of the general reasons for such illiquidity risks is that banks tend to roll-over loans, as short-term loans come at a lower interest rate than medium length loans. A bank becomes illiquid if the overnight money market dries up, and it cannot roll-over its loans fast enough to fulfill its liabilities. In such events banks may rely on government bailouts or liquidation of less liquid position at fire-sale prices (i.e. Asset liquidity risk due to urgency). Recent bank runs have shown that funding illiquidity can lead to serious financial problems and high interest rates for short loans. Since the crisis of 2007 the Basel committee has increased regulatory guidelines for liquidity risk.

Both types of liquidity risks discussed here are difficult to model in practice, as they apply to distressed market conditions for which only few models are in place. There are
however a few measures that can be used to get a feeling for the liquidity risk exposure a company has due to a specific position:

**Bid-Ask spreads**  Bid-ask spreads can be used to get a grip on the depth of the market. Models that combine bid-ask spreads and resiliency could be used to determine an optimal buying/selling strategy to obtain/liquidate a position. (When markets are thin it is not optimal to buy/sell the position in one order).

**Resilience**  Resilience is the recovery of the market price after a large order has been placed. Large orders tend to increase the bid-ask spread, especially in thin markets. A market is said to be resilient if its recovery rate is high.

**Position size**  Unlike bid-ask spreads and resilience, position size is a property of the company opposed to a property of the market. Therefore, liquidity risks originating from position size are endogenous while the other two are exogenous. Liquidity risk increases as the size of the position a company holds relative to the total size of the market is large. In such cases it is unlikely that the company can liquidate its position at the fair market price easily.

**A.2.4 Operational risk**

Operational Risk is different from forms of risk we have seen before as it originates from within the firm itself, hence the name operational risk. Examples of operational risk include management failure, faulty controls, fraud and human error. Depending on the type of business a company is in, operational risk can be a serious source of risk, which requires tight controls. The banking industry can easily be exposed to operational risks due to wrong valuations in derivative trading (albeit intentional, i.e. fraud, or human error.)

The value of modeling operational risk has only recently been acknowledged, and as such modeling these risks is lagging behind other risk areas. Nevertheless, models are being developed, and this is certainly a form of risk for which economic scenario generation can prove valuable.

**A.2.5 Legal and regulatory risk**

These types of Risk may occur due to at least two different reasons: first, a counter party may decide to sue the company rather than meet its financial obligations. The
resulting legal struggle may impose financial consequences on the company. Second, changes in law or regulations may lead to specific contracts dropping in value. Both types of risk are very hard to anticipate, and as such very little effort has been put into modeling such risks. The common practice is to simply reserve an amount that can be written off in case such Risks present themselves.

A.2.6 Business risk

Business risk are all those risks that arise from regular business practice. In a production setting, it concerns day-to-day sales price fluctuations, changes in base cost of goods, and uncertainty about demand. In the banking industry however, such risks are hard to differentiate from already captured market risk, credit risk, and operational risks. It is for this reason that in the banking industry, business risk is often combined with strategic and reputation risk into one header.

A.2.7 Strategic risk

Strategic risks arise from significant but uncertain strategic investments, such as takeovers and mergers. Such investments may lead to large write offs, possibly leading to the default of the parent company. Such possible extreme negative effects need to be taken into account when valuing the investment.

A.2.8 Reputation risk

Like strategic risk, reputation risk is hard to quantify. However, since the Enron scandal a large (34%) portion of bankers believe that reputation risk is the biggest risk in the banking industry. Since the scandals of the 1990s regulators are focusing more on these hard to quantify risks, as they have proven to be genuinely troublesome. Possibly even more so in the banking industry, as the confidence and in recent years also the approval of customers is crucial for a prospective future.

Again, reputation risk, like business- and strategic risk, is very hard to measure and therefore to reserve capital for. In the case of reputation risk this issue is magnified: reserving capital for reputation risk in a way equals stating that the entity expects to take a reputation hit in the future, and thus immediately endangers its own reputation.
A.3 Applications of economic scenario generators

In this section we will consider possible applications of ESG models. Again, we will focus on applications in the banking industry.

A.3.1 Credit-value adjustment

Credit Value Adjustments (CVA) are value adjustments on financial contracts, made based on the creditworthiness of a counter party. Such adjustments depend heavily on the PD and LGD of the counter party, but other factors may be relevant as well. As mentioned earlier, both PD and LGD are hard to determine. ESG models can be helpful, as correctly calibrated models can give an insight in the default chances, and possibly the distribution of the LGD figure. Key challenges when using ESG models for CVA are the following:

- Market Consistent (risk neutral) Calibration;
- Mark-to-market values;
- Correctly modeling and calibrating interest rates is crucial as most CVA instruments are taken on swaptions;
- CVA deals with defaulting companies, hence a default model must be implemented. Default models are subjective; and
- Pricing of complex derivatives.

A.3.2 Assets & liabilities management

Asset and Liability Management (ALM) is a difficult task in both the banking and insurance industries. ESG models for ALM also need to value some hard to model assets, such as mortgages. Modeling such assets is usually done through behavioral models, as the value of outstanding assets is heavily influenced by client behavior such as early repayment of mortgages. These behavioral models are subjective and little is known about this proper modeling of future human financial behavior. Also, since the entire assets portfolio needs to be considered at once, the model does need to take correlations between equities into account. Such correlations are hard to estimate as very little is known about such figures in finance. A third challenge lies in modeling long-term interest rates. Where CVA contracts typically have a duration of no more than twenty years, mortgages are known to run for up to sixty. Modeling, and more
importantly calibration of said models for long-term interest rates is notoriously difficult. Summarizing we see the following challenges if we consider ALM as goal for our ESG:

- Market Consistent (risk neutral) Calibration;
- Behavioral models needed for valuation of assets;
- Correlations of equity portfolios are hard to estimate from market data;
- Modeling of long-term interest rates; and
- Pricing of complex derivatives.

A.3.3 Stress testing

Stress testing is an increasingly popular tool in the banking industry. It can be used to identify weak areas to which banks are exposed, and situations in which banks go into default. Traditional stress testing leads to either a pass or fail for a specific scenario. Another approach is reverse stress testing, with the main goal of finding the crucial point after which a company goes into default.

Other than testing for default, stress testing can also be applied to parts, such as departments of a company, to determine its performance under distress. Depending on the size and complexity of the (part of) the company being tested, stress testing can become very computationally intensive. It is for this reason that stress testing is relatively new, however testing a complete bank under a whole range of scenarios still remains unviable today. Note that stress testing may include CVA calculations under different scenarios, which are already known to require multiple days of computations if done thoroughly.

The value of stress tests depends on the choice of the scenarios. Realistic scenarios, possibly coupled with probability figures, give far more valuable output than randomly chosen scenarios. Especially in the case of large companies it may only be possible to test several scenarios in the available time frame, and as such choosing scenarios smartly is a serious challenge. Stress testing is an interesting topic for this research, and would have the following characteristics:

- Risk Neutral world;
- Probability spectrum of scenarios;
- Simulation of multiple, correlated equities;
• Generation of shocks and outliers (truly stressful scenarios);

• Highly Technical topic; and

• Identified as one of the most challenging topics in the field of mathematical finance.

A.3.4 Illiquidity

Modeling illiquidity is a relatively new area, where financial institutions wish to account for liquidity risk. In the recent crisis of 2007 multiple sound firms have gotten into significant problems because the short-term funding market had dried up completely. Before the events of 2007 researchers and regulators had limited interest in modeling liquidity risk, but the value of liquidity models has become clear since then. However, research in this area has progressed significantly less far than in areas such as CVA and ALM. It will therefore be challenging to find models in the literature that can be used for the purpose of economic scenario generation. First efforts in modeling liquidity risk are focused on combinations of partial differential equations and feedback loops. Feedback loops may be used to overcome a problem with correlations: correlating liquidity and asset prices is not trivially done in a way that is consistent with reality. Rather than relying solely on SDEs such as in Equity models, recent research is focused on combining such SDEs with a process that in turn triggers a feedback loop. More research would need to be done if we choose this direction.

In more advanced stages of liquidity modeling we would also be interested in the behavior of the long-term interest rate as the short-term interest rate spikes. This requires complex models for the long-term rate, but unlike in ALM the calibration is not crucial as no valuations are being made based on the model. Even so, modeling and correctly correlating it with the short rate and liquidity process would be one of the challenges.

• Limited theory;

• Requires implementation of a complex long-term interest rate model; and

• Correlating liquidity and asset price may be troublesome as no clear relation exists.
Appendix B

Choice for Nelson-Siegel Interpolation

In section 5.4 we described interpolation methods, following the research by Hagan [6]. We concluded that the choice for a monotone convex method would be best, but that we would use cubic spline interpolation as a starting point. We have however investigated the implications of this choice, and found that cubic spline interpolation is a very crude method when applied for yield-curve construction. The method is hardly able to replicate the initial term structure, because it does not take into account the first derivative in the data points. Especially at the first and last data points, this causes the curve to be directed wrong. (For instance, we know the yield curve is asymptotic for long maturities, something the CS method fails to realize.) In retrospect this should not have surprised us, as this method simply uses the least squares approach fitting a polynomial through the data. Since the method is not able to fit the initial curve it would be quite wrong to use it to describe observed changes in shape due to our modifications.

Nelson Siegel interpolation on the other hand is of the monotone convex form, and is used widely for yield-curve calibration in the industry. When this method is applied we do retrieve the typical yield-curve shapes we also expect to see in real market situations. Initially we expected CS interpolation to perform ’well enough’ for comparison purposes, but after implementing both method the choice for NS interpolation was definitive. Figure B.1 displays the curves obtained with NS interpolation. The results of CS interpolation on the same data is displayed in B.2, visual comparison should clarify our choice for NS interpolation further.
Appendix B. Choice for Nelson-Siegel Interpolation

Figure B.1: Yield curves for unmodified data with Nelson Siegel interpolation.

Figure B.2: Yield curves for unmodified data with Cubic Spline interpolation.
Appendix C

Derivation of Closed-Form Solutions

C.1 Solution to linear-percentage impact model

Derivation of the solution to this model is shown through induction on $k$. First, recall the control laws are as follows:

$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \epsilon_t, \quad \theta > 0 \tag{C.1}$$
$$X_t = \rho X_{t-1} + \eta_t, \quad \rho \in (-1, 1) \tag{C.2}$$

We wish to show that the optimal control policy is as in C.3-C.4

$$S^*_T = \delta_{w,k} W_T + \delta_{x,k} X_T \tag{C.3}$$
$$V_T(P_{T-k-1}, X_{T-k}, W_{T-k}) = P_{T-k-1} W_{T-k} + a_k W_{T-k}^2 + b_k X_{T-k} W_{T-k} + c_k X_{T-k}^2 + d_k \tag{C.4}$$

where

$$\delta_{w,k} = \frac{1}{k+1}, \quad \delta_{x,k} = \frac{\rho b_{k-1}}{2a_{k-1}}$$

To this end, we start the induction with $k = 0$. Then the optimal control $S^*_T = W_T$, and the value function becomes as in C.5, after substituting $a_0 = \theta$, $b_0 = \gamma$, $c_0 = 0$ and $d_0 = 0$.

$$V_T(P_{T-1}, X_T, W_T) \equiv \min_{S_T} \mathbb{E}_T [P_T S_T] = (P_{T-1} + \theta S_T + \gamma X_T) S_T \tag{C.5}$$
Now, assume that C.4 holds for $k$, we prove it also holds for $k+1$ by applying the Bellman equation.

$$V_{T-k-1}(P_{T-k-2}, X_{T-k-1}, W_{T-k-1}) = \min_{S_{T-k-1}} E_{T-k-1}[P_{T-k-1} + V_{T-k}(P_{T-k-1}, X_{T-k}, W_{T-k})]$$

We now substitute equation C.4 into equation C.6

$$V_{T-k-1}(P_{T-k-2}, X_{T-k-1}, W_{T-k-1}) = \min_{S_{T-k-1}} E_{T-k-1}\left[P_{T-k-1} + P_{T-k-1}W_{T-k} + a_k W_{T-k}^2 + b_k X_{T-k}W_{T-k} + c_k X_{T-k}^2 + d_k\right]$$

Now, filling in equations C.1-C.2 in equation C.7:

$$V_{T-k-1}(P_{T-k-2}, X_{T-k-1}, W_{T-k-1}) = \min_{S_{T-k-1}} \left[(P_{T-k-2} + \theta S_{T-k-1} + \gamma X_{T-k-1})W_{T-k-1} + a_k(S_{T-k-1} - W_{T-k-1})^2 + b_k \rho X_{T-k-1}(W_{T-k-1} - S_{T-k-1}) + c_k(\rho^2 X_{T-k-1}^2 + \sigma_\eta^2 + d_k)\right]$$

Since the minimization in C.8 is convex because $a_k > 0$, its optimal solution is as follows:

$$S_{T-k-1}^* = \left(1 - \frac{\theta}{2a_k}\right)W_{T-k-1} + \frac{\rho b_k}{2a_k} X_{T-k-1}$$

Equation C.9 reduces to equation C.4 when we insert the definition of $V_i(P_{t-1}, W_i) = \min_{S_i} E_i[P_t S_t + V_{t+1}(P_t, W_{T+1})]$. Now, observe that $a_k$ satisfies C.10:

$$a_k = \theta \left(1 - \frac{\theta}{4a_{k-1}}\right), \quad a_0 = \theta$$

Which is solved by

$$a_k = \frac{\theta}{2} \left(1 + \frac{1}{k+1}\right)$$

Such that we can write equation C.12, and complete the proof of the solution.

$$S_{T-k-1}^* = \frac{W_{T-k-1}}{k} + \frac{\rho b_k}{2a_k} X_{T-k-1}$$
C.2 Solution to linear-percentage temp-impact model

Here too, induction on \( k \) is used to derive the solution. First recall the control laws of the linear-percentage temp-impact model:

\[
P_t = \tilde{P}_t + \Delta_t \tag{C.13}
\]
\[
\tilde{P}_t = \tilde{P}_{t-1} \exp(Z_t) \tag{C.14}
\]
\[
\Delta_t = (\theta S_t + \gamma X_t) \tilde{P}_t \tag{C.15}
\]
\[
X_t = \rho X_{t-1} + \eta_t \tag{C.16}
\]

Again, we write out \( V_T \) and start the recursion from there:

\[
V_T(P_{T-1}, X_{T-1}, W_T) = \min_{S_T} \mathbb{E}_T[P_T S_T] = \mathbb{E}_T[P_T W_T] \tag{C.17}
\]
\[
= q P_{T-1} (W_T + \gamma X_T W_T + W_T^2) \tag{C.18}
\]

where \( q \equiv \mathbb{E}_T[\exp(Z_T)] = \mathbb{E}[\exp(Z_T)] = \exp(\mu_z + \frac{\sigma_z^2}{2}) \tag{C.19} \)

Now, in period \( T - 1 \) the optimal value function becomes:

\[
V_{T-1} = \min_{S_{T-1}} \mathbb{E}_{T-1}[P_{T-1} S_{T-1} + V_{T}(P_{T-1}, X_{T-1}, W_T)] \tag{C.20}
\]
\[
= \min_{S_{T-1}} \{ P_{T-1} (1 + \theta S_{T-1} + \gamma X_{T-1}) S_{T-1} + q P_{T-1} \}
\times (W_{T-1} - S_{T-1}) [1 + \gamma \rho X_{T-1} + \theta (W_{T-1} - S_{T-1})] \} \tag{C.21}
\]

The optimal policy at \( T - 1 \) then becomes:

\[
S^*_{T-1} = \delta_{x,1} X_{T-1} + \delta_{w,1} W_{T-1} + \delta_{1,1} \tag{C.22}
\]

with \( \delta_{x,1} = \frac{\gamma (q \rho - 1)}{2 \theta (q+1)}, \quad \delta_{w,1} = \frac{q}{q+1}, \quad \delta_{1,1} = \frac{q - 1}{2 \theta (q+1)} \tag{C.23} \)
And thus the optimal value function at \( T - 1 \) is equal to:

\[
V_{T-1}(P_{T-2}, X_{T-1}, W_{T-1}) = qP_{T-2}[a_1 + b_1X_{T-1} + c_1X^2_{T-1} + d_1X_{T-1}W_{T-1} + e_1W_{T-1} + f_1W^2_{T-1}]
\]

where

\[
a_1 = \delta_{1,1}(1 + \theta\delta_{1,1}) - q\delta_{1,1}(1 - \theta\delta_{1,1}),
\]

\[
b_1 = (1 - q)\delta_{x,1},
\]

\[
c_1 = \delta_{x,1}(\theta\delta_{x,1} + \gamma) - q\delta_{x,1}(\gamma\rho - \theta\delta_{x,1}),
\]

\[
d_1 = \gamma(1 + \rho)\delta_{w,1},
\]

\[
e_1 = 2\delta_{w,1},
\]

\[
f_1 = \theta\delta_{w,1}
\]

This procedure yields equation C.31 for period \( T - k \):

\[
S_{T-k} = \delta_{x,k}X_{T-k} + \delta_{w,k}W_{T-k} + \delta_{1,k}
\]

where

\[
\delta_{x,k} = \frac{q \sigma_{k-1} - \gamma}{2(\theta + qf_{k-1})}, \quad \delta_{w,k} = \frac{q \rho_{k-1}}{\theta + qf_{k-1}}, \quad \delta_{q,k} = \frac{qe_{k-1} - 1}{2(\theta + qf_{k-1})}
\]

We can now expand the value function, to obtain equations C.33-C.39:

\[
V_{T-k}(P_{T-k-1}, X_{T-k}, W_{T-k}) = qP_{T-k-1}[a_k + b_kX_{T-k} + c_kX^2_{T-k} + d_kX_{T-k}W_{T-k} + e_kW_{T-k} + f_kW^2_{T-k}]
\]

where

\[
a_k = \delta_{1,k}(1 + \theta\delta_{1,k}) + q(a_{k-1} + \sigma^2_{k-1}) - q\delta_{1,k}(e_{k-1} - \delta_{1,k}f_{k-1}),
\]

\[
b_k = q\rho b_{k-1} - \delta_{x,k}(qe_{k-1} - 1),
\]

\[
c_k = \delta_{x,k}(\theta\delta_{x,k} + \gamma) + q\rho^2 c_{k-1} - q\delta_{x,k}(\rho\sigma_{k-1} - \delta_{x,k}f_{k-1}),
\]

\[
d_k = \gamma\delta_{w,k} + q\rho d_{k-1}(1 - \delta_{w,k}),
\]

\[
e_k = \delta_{w,k} + q(1 - \delta_{w,k})e_{k-1},
\]

\[
f_k = \theta\delta_{w,k}
\]

Which completes the proof of the solution.
Appendix D

Calibration Procedure for Direct Simulation Methods

The following procedure can be followed to calibrate the 'direct simulation on interest products' method. Note that in essence it is very similar to the recursive learning methods discussed for both Hull White families of models.

- Find an initial strategy based on a first conservative guess for the liquidity dependency parameters.
- Execute the first order in the strategy.
- Observe the liquidity premium and compare to the price that was expected under the strategy.
- Vary model parameters until a strategy is found that matches the yield-curve shift that was observed.
- Iterate as more orders are executed until the model prediction matches reality satisfactory.
Bibliography


